A Jump-Diffusion Nominal Short Rate Model*

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Abstract

This paper extends the monetary equilibrium approach of Lioui and Poncet (2004) to a jump-diffusion setting. We show that in the presence of jumps money non-neutrality is preserved and the jump component of the inflation risk premium is affected, in addition to technology factors, by monetary policy variables. Finally, we derive the jump-diffusion dynamics of a nominal short interest rate.

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Modeling interest rates has been achieved through two different approaches : the arbitrage-free framework[1] and the general equilibrium approach. The equilibrium setting is very useful to gain insights into the economic background of the model and to understand the interplay between variables. In addition, equilibrium models explicitly specify the risk premia.
In the finance literature, one of the most used general equilibrium framework is the seminal work of Cox et al. (1985a,b) (hereafter CIR) who derived in a real production economy, the dynamics of the short-term real interest rate. Therefore, using a nominal interest rate model, based on the results of CIR, assumes a stringent condition on the inflation rate. That is, the inflation rate process is assumed to be non stochastic.

To relax this assumption, Lioui and Poncet (2004) (hereafter LP) built a general equilibrium model where a monetary economy affected by both real and monetary shocks is considered. Relying on the money-in-utility function where the investor holds money in addition to consumption, LP investigate money non-neutrality and show that money cannot be neutral due to the important role monetary parameters play within various financial quantities. In addition to specifying the endogenous expressions for the drift and diffusion terms of the general price level, LP focus on deriving, under specific assumptions, different dynamics of the real and nominal interest rates. Nevertheless, since LP consider the case of pure diffusion processes, the obtained dynamics of the various short-term interest rates do not meet empirical evidence on the presence of discontinuities in the process of the interest rate. For instance, Das (2002) and Johannes (2004) provide evidence of the presence of jumps in the dynamics of interest rates. Similarly, Guan et al. (2005) find evidence that a jump-diffusion setting is required to efficiently explain the Libor rates dynamics.

The goal of this paper is to derive a jump-diffusion model for the nominal interest rate based on economic arguments. This is achieved by extending the monetary economy of LP to a jump-diffusion setting. Another consequence of this extension is that the inflation rate process incorporates jumps, in addition to its diffusion component. This feature is very likely since the inflation rate can be subject to jumps on macroeconomic data releases. Abrupt adjustments in the inflation rate may occur when the realized rate is different from the expected one[2]. We also analyze the impact of introducing jumps on various

[1] See, for instance, Black et al. (1990), Heath et al. (1992) and Brace et al. (1997).

[2] For instance, one can read in a major media network : *Britain's inflation rate picked up more than expected in August, boosted by strong rises in toys and computer games[...]. Consumer prices rose by 0.4 percent last month, taking the annual rate up to 2.5 percent [...]. The data was above analysts' forecasts for a rate of 2.4 percent.* (Excerpt from a September 12, 2006 article from cnn.com).

financial entities such as the expected excess return on a nominal bond and the inflation risk premium.

This article is organized as follows. Section (1) presents the general setting and derives the dynamics of the general price level. Assuming, for tractability, a log separable utility function, we also obtain expressions for the real and nominal short term interest rates as well as the inflation risk premium. Section (2) derives, under specific assumptions for the dynamics of the state variables and the production and monetary supply, the dynamics of the nominal short-term interest rate. Section (3) concludes.

1 The economy

We consider, as in Lioui and Poncet (2004), an economy with real and monetary sectors. In the real sector, there is a single good, which may be allocated to either consumption or investment, produced by a single technology. In the monetary sector, the Central Bank, based on economic targets, sets the money supply. We, therefore, assume an exogenous money supply process.

We differ from the framework of LP by introducing jumps in the dynamics of all the processes involved. That is, the production, money supply and the state variables follow jump-diffusion processes.

We assume the existence of K state variables which dynamics obey to the system of the stochastic differential equations (SDE) below :

$$dY(t) = \mu(t, Y)dt + \sigma(t, Y)'dZ(t) + \gamma(t-, Y)'dN(t) \tag{1}$$

where $\mu(t, Y)$ is a bounded $K \times 1$ vector function of drifts, $\sigma(t, Y)$ is a bounded $K \times (N + K)$ diffusion matrix function and $\gamma(t, Y)$ is a bounded $K \times (N + K)$ matrix function of independent jump amplitudes. $N(t)$ is the $(N + K) \times 1$ vector of independent Poisson processes with $(K + N) \times 1$ vector of constant intensity λ. $Z(t)$ is an $(N + K) \times 1$ vector of independent Wiener processes. The Poisson and Wiener processes are defined on the complete filtered probability space $(\Omega, \mathbb{F}, \mathbf{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ and $\mathbf{P}$ the historical probability measure.

The production of the good is governed by the SDE below :

$$\frac{d\eta(t)}{\eta(t-)} = \mu_\eta(t, Y)dt + \sigma_\eta(t, Y)'dZ(t) + \gamma_\eta(t, Y)'dN(t) \tag{2}$$

where $\eta(t)$ denotes the amount of the good invested at date t in the technology., $\mu_\eta(t,Y)$ is a bounded function, $\sigma_\eta(t,Y)$ is a bounded $(N+K)\times 1$ vector valued function, and $\gamma_\eta(t,Y)$ is $(N+K)\times 1$ vector valued function.

Moreover, the money $M(t)$ issued by the Central Bank has the following dynamics :

$$\frac{dM(t)}{M(t-)} = \mu_M(t,Y)dt + \sigma_M(t,Y)'dZ(t) + \gamma_M(t,Y)'dN(t) \tag{3}$$

where $\mu_M(t,Y)$ is a bounded function and $\sigma_M(t,Y)$ and $\gamma_M(t,Y)$ are two $(N+K)\times 1$ vector valued functions.

It will be shown that the general price level, $P(t)$, expressed in units of the consumption/investment good, obeys to the dynamics below

$$\frac{dP(t)}{P(t-)} = \mu_p(t,Y)dt + \sigma_p(t,Y)'dZ(t) + \gamma_p(t,Y)'dN(t) \tag{4}$$

where $\mu_p(t,Y)$ is a bounded function and $\sigma_p(t,Y)$ and $\gamma_p(t,Y)$ are two $(N+K)\times 1$ vector valued functions. The quantities $\mu_p(t,Y)$, $\sigma_p(t,Y)$ and $\gamma_p(t,Y)$ will be determined endogenously.

Furthermore, we assume the existence of a competitive market for a real default-free zero-coupon bond $b(t,T,w,Y)$ which dynamics[3] are :

$$\frac{db(t,T)}{b(t-,T)} = \mu_b(t,Y,T)dt + \sigma_b(t,Y,T)'dZ(t) + \gamma_b(t,Y,T)'dN(t) \tag{5}$$

where $\mu_b(t,Y,T)$ is the expected rate of return on the real zero-coupon bond and $\sigma_b(t,Y,T)$ and $\gamma_b(t,Y,T)$ are two $(N+K)\times 1$ vectors. Applying Itô's lemma on the real zero-coupon bond price, one can determine the values of $\sigma_b(t,Y,T)$ and $\gamma_b(t,Y,T)$ as follows :

$$\sigma_b = \frac{b_w}{b}\sigma_w + \frac{b_Y}{b}\sigma \tag{6}$$

$$\gamma_b = \frac{1}{b}[b(W+\gamma_w, Y+\gamma, t, T) - b(t,T)]' \tag{7}$$

μ_b will be determined endogenously in the sequel. In addition, it is assumed the existence of real and nominal money market accounts. The real one has a riskless, in real terms, instantaneous real return equal to the real interest rate $r(t)$. The nominal savings, which

[3] For clarity of exposition, we drop the dependence on the state variables Y and the aggregate real wealth w.

is also riskless but in nominal terms[4], has an instantaneous nominal return equal to the nominal interest rate $R(t)$. Of course, since the bond gives unity at maturity regardless of the real wealth, we can delete from Eqs. (6) and (7) the dependence on w.

We finally assume that continuous trading in all financial and monetary assets and in the technology takes place in frictionless and arbitrage-free markets and at equilibrium prices only.

We consider a representative agent that holds, at time t, a number $\kappa(t)$ of units of the technology which real value is $\kappa(t)\eta(t)$. The ratio investment to real wealth is denoted by $\alpha(t) \equiv \frac{\kappa(t)\eta(t)}{w(t)}$, where $w(t)$ is the investor's real wealth. The representative investor aims to maximize, over an infinite horizon, his or her expected utility function subject to the wealth constraint :

$$\max_{c,m,\alpha,\theta,\delta} \mathbb{E}\left(\int_0^\infty U(t, c_t, m_t) dt \right) \tag{8}$$

where $c(t)$ is the time t consumption rate and $m(t)$ denotes real money balances holdings. $U(t, c_t, m_t)$ is the standard money-in-the-utility function.

For ease of exposition, the explicit dependence of the variables on time and state variables are dropped out. The investor's budget constraint is given by :

$$\begin{aligned} dw = \Big[& \alpha w(\mu_\eta - r) + \theta w(\mu_b - r) + \delta w(R - (\mu_p - \sigma_p' \sigma_p) - r) + wr \\ & - c - m(\mu_p - \sigma_p' \sigma_p + r) \Big] dt + \Big[\alpha w \sigma_\eta + \theta w \sigma_b - (\delta w + m)\sigma_p \Big] dZ(t) \\ & + \Big[\alpha w \gamma_\eta + \theta w \gamma_b - (\delta w + m) \frac{\gamma_p}{1+\gamma_p} \Big] dN(t) \end{aligned} \tag{9}$$

where θ and δ represent proportions of real wealth invested respectively in the discount bonds and the nominal money market account.

The equilibrium of the economy is characterized by the market clearing conditions :

- total wealth must be equal to the total amount invested in the technology plus the holdings of real value money balances : $\kappa\eta + m = w$,
- net holdings in the nominal money market accounts and in each of the various contingent claims must be nil, i.e. $\delta = 0$ and $\theta = \mathbf{0}$
- money supply must equal money demand, i.e. $\dfrac{M}{P} = m = w(1 - \alpha)$.

[4]The nominal money market account is a risky asset in real terms.

Under the above equilibrium conditions, Eq.(9) becomes

$$\begin{aligned}\frac{dw}{w} &= \left[\alpha\mu_\eta - \frac{c}{w} - (1-\alpha)(\mu_p - \sigma_p'\sigma_p)\right] dt \\ &+ \left[\alpha\sigma_\eta - (1-\alpha)\sigma_p\right] dZ(t) + \left[\alpha\gamma_\eta - (1-\alpha)\frac{\gamma_p}{1+\gamma_p}\right] dN(t)\end{aligned} \tag{10}$$

In addition, given that $P = \dfrac{M}{m}$, applying the generalized Ito's lemma yields :

$$\begin{aligned}\frac{dP_t}{P_{t-}} &= \left[\mu_M - \mu_m + \sigma_m'\sigma_m - \sigma_M'\sigma_m\right] dt \\ &+ \left[\sigma_M - \sigma_m\right] dZ(t) + \left[\gamma_M - \frac{\gamma_m}{1+\gamma_m}(\gamma_M + 1)\right] dN(t)\end{aligned} \tag{11}$$

One can, straightforwardly, obtain

Proposition 1 *At equilibrium, the dynamics of the price level and the wealth are*

$$\frac{dP(t)}{P(t-)} = \mu_p dt + \sigma_p dZ(t) + \gamma_p dN(t) \tag{12}$$

where

$$\mu_p = \frac{1}{1-\alpha}\mu_\alpha + \frac{1}{\alpha}\mu_M + \frac{c}{w\alpha} - \mu_\eta + \frac{\alpha}{(1-\alpha)^2}\sigma_\alpha'\sigma_\alpha + \sigma_\eta'\sigma_\eta - \frac{1}{\alpha}\sigma_M'\sigma_\eta + \frac{1}{1-\alpha}\sigma_M'\sigma_\alpha - \frac{1}{1-\alpha}\sigma_\alpha'\sigma_\eta \tag{13}$$

$$\sigma_p = \frac{1}{1-\alpha}\sigma_\alpha + \frac{1}{\alpha}\sigma_M - \sigma_\eta \tag{14}$$

$$\gamma_p = \frac{1}{1+\gamma_\eta}\left[\frac{\gamma_\alpha}{1-\alpha(1+\gamma_\alpha)} + \frac{1-\alpha}{\alpha}\frac{\gamma_M}{1-\alpha(1+\gamma_\alpha)} - \gamma_\eta\right] \tag{15}$$

and

$$\frac{dw(t)}{w(t-)} = \mu_w dt + \sigma_w dZ(t) + \gamma_w dN(t) \tag{16}$$

where

$$\mu_w = \mu_\eta - \frac{c}{\alpha w} - \mu_\alpha - \frac{1-\alpha}{\alpha}\mu_M + \sigma'_\alpha\sigma_\alpha - \frac{1-\alpha}{\alpha}\sigma'_M\sigma_\eta + \frac{2-\alpha}{\alpha}\sigma'_M\sigma_\alpha + \frac{1-\alpha}{\alpha^2}\sigma'_M\sigma_M - \sigma'_\alpha\sigma_\eta \tag{17}$$

$$\sigma_w = \sigma_\eta - \sigma_\alpha - \frac{1-\alpha}{\alpha}\sigma_M \tag{18}$$

$$\gamma_w = \frac{\alpha\gamma_\eta - \alpha\gamma_\alpha + \alpha\gamma'_M\gamma_\eta - (1-\alpha)\gamma_M}{\gamma_M + \alpha(1+\gamma_\alpha)} \tag{19}$$

Eqs.(13), (14), (17) and (18) are similar to those obtained by LP in the pure diffusion case. We, therefore, focus our attention to the discussion of the jump related equations, that is, Eqs. (15) and (19). First, one can notice that the jump component in the wealth dynamics is affected by both monetary and real parameters. Hence, money non-neutrality is preserved in the presence of jumps.
Furthermore, Eq. (15) shows that the jump amplitude in the inflation index rate has the same sign as that in the money supply.

1.1 Equilibrium interest rates

In order to obtain the expression of the equilibrium real and nominal interest rates as well as the expected return of the discount bond, we first rely on the first-order conditions. To this end, let us consider $J(t,W,Y)$ be the indirect utility function which satisfies the HJB jump-diffusion equation

$$0 = \max(U + J_w w\mu_w + \frac{1}{2}J_{ww}w^2\sigma_w^2 + J_Y\mu + \frac{1}{2}J_{YY}\sigma^2 + J_{wY}\sigma_w\sigma + \lambda\mathbb{E}[J(t, W+\gamma_w, Y+\gamma) - J(t,W,Y)]) \tag{20}$$

Define $\Psi \equiv \mathcal{L}J + U$, where $\mathcal{L}J$ is the differential generator of J. The first-order conditions for the consumer's optimization are

$$\Psi_c = U_c - J_w \leq 0 \tag{21}$$

$$c\Psi_c = 0 \tag{22}$$

$$\begin{aligned}\Psi_m &= U_m - J_w(\mu_p - \sigma_p^2 + r) + J_{ww}(m\sigma_p^2 + \delta w\sigma_p^2 - \theta w\sigma_b\sigma_p - \alpha w\sigma_\eta\sigma_p) \\ &\quad - J_{wY}\sigma_p\sigma - \lambda\mathbb{E}\big[\frac{J_w(w+\gamma_w, Y+\gamma)}{w}\frac{\gamma_p}{1+\gamma_p}\big] \leq 0\end{aligned} \tag{23}$$

$$m\Psi_m = 0 \tag{24}$$

$$\begin{aligned}\Psi_\alpha &= J_w w(\mu_\eta - r) + J_{ww}(\alpha w^2\sigma_\eta^2 + \theta w^2\sigma_\eta\sigma_b - \delta w^2\sigma_p\sigma_\eta - mw\sigma_p\sigma_\eta) \\ &\quad + wJ_{wY}\sigma_\eta\sigma + \lambda\mathbb{E}\big[J_w(w+\gamma_w, Y+\gamma)\gamma_\eta\big] \leq 0\end{aligned} \tag{25}$$

$$\alpha\Psi_\alpha = 0 \tag{26}$$

$$\begin{aligned}\Psi_\delta &= J_w w(R - (\mu_p - \sigma_p^2) - r) + \tfrac{1}{2}J_{ww}(2\delta w^2\sigma_p^2 + 2mw\sigma_p^2 - 2w^2\alpha\sigma_p\sigma_\eta - 2w^2\sigma_p\theta\sigma_b) \\ &\quad - wJ_{wY}\sigma_p\sigma - \lambda\mathbb{E}\big[J_w(w+\gamma_w, Y+\gamma)\frac{\gamma_p}{1+\gamma_p}\big] = 0\end{aligned} \tag{27}$$

$$\begin{aligned}\Psi_\theta &= J_w w(\mu_b - r) + J_{ww}(\theta w^2\sigma_b^2 + \alpha w^2\sigma_\eta\sigma_b - \delta w^2\sigma_p\sigma_b - mw\sigma_p\sigma_b) \\ &\quad + wJ_{wY}\sigma_b\sigma + \lambda\mathbb{E}\big[J_w(w+\gamma_w, Y+\gamma)\gamma_b\big] = 0\end{aligned} \tag{28}$$

Explicit expressions can be obtained if a specific form of the utility function is provided. For tractability, we assume, as in LP, a log-separable utility function :

$$U(t, c_t, m_t) = e^{-\rho t}\big(\beta\log(c_t) + (1-\beta)\log(m_t)\big) \tag{29}$$

where ρ is the rate of impatience and β is a weighting factor. The indirect utility function is assumed to be state independent and has the following form :

$$J(t, w(t)) = \frac{1}{\rho}e^{-\rho t}\log(Aw(t)) \tag{30}$$

With Eqs.(29) and (30), the explicit real and nominal interest rates are given by the following proposition :

Proposition 2 *At equilibrium and given (29) and (30) the real and nominal interest rates* *are given by*

$$r = \mu_\eta - \sigma_\eta^2 + \sigma_\alpha\sigma_\eta + \frac{1-\alpha}{\alpha}\sigma_M\sigma_\eta + \lambda\mathbb{E}\big[\frac{\gamma_\eta\gamma_M}{\alpha(1+\gamma_\eta)(1+\gamma_M)} + \frac{1}{1+\gamma_M} + \frac{\gamma_\eta\gamma_\alpha - 1}{(1+\gamma_\eta)(1+\gamma_M)}\big] \tag{31}$$

$$R = \rho + \frac{\alpha}{1-\alpha}\mu_\alpha + \mu_M - \sigma_M^2 - \frac{\alpha}{1-\alpha}\sigma_\alpha\sigma_M + \lambda\mathbb{E}\big[\frac{\gamma_M}{1+\gamma_M} + \frac{\alpha}{1-\alpha}\frac{\gamma_\alpha}{1+\gamma_M}\big] \tag{32}$$

respectively.

With respect to the results of LP in the pure diffusion case, the expressions (31) and (32) contain additional jump related terms. They are the fifth and sixth terms in Eqs. (31) and (32), respectively. We reach, when introducing jumps, similar conclusions as in the pure diffusion case, that is, the real interest rate is affected by monetary and technological parameters whereas the nominal interest rate is solely determined by monetary factors[5]. We can also point out that the jump component in the nominal interest rate is caused by monetary policy events. This result is very intuitive since the Central Bank sets, periodically, the level of the short-term nominal rate.

Proposition 3 *At equilibrium, the expected excess return on a real discount bond is given by*

$$\mu_b - r = \sigma_\eta\sigma_b - \sigma_\alpha\sigma_b - \frac{1-\alpha}{\alpha}\sigma_M\sigma_b - \lambda\mathbb{E}\left[\frac{\gamma_b}{1+\gamma_w}\right] \tag{33}$$

The expected excess return depends, in addition to the covariance between the bond volatility and the real wealth one, on the sign of jump amplitude of the bond price. Specifically, for negative jumps, the expected excess return increases and inversely. This is a well know result : when the bond price rises its yield decreases.

The nominal discount bond, B, can, therefore, be obtained by applying Ito's lemma to $B = bP$. Thus, the nominal expected return of the discount bond is given by

$$\mu_B = R + \sigma_B\big(\frac{\alpha}{1-\alpha}\sigma_\alpha + \frac{1}{\alpha}\sigma_M\big) - \lambda\mathbb{E}\left[\frac{1-\alpha(1+\gamma_\alpha)}{(1-\alpha)(1+\gamma_M)}\gamma_B\right] \tag{34}$$

where $\sigma_B = \sigma_b + \sigma_p$ and $\gamma_B = \gamma_b(1+\gamma_p) + \gamma_p$. As in the case of the real bond, the expected excess return on the nominal bond decreases when jumps in the bond price are positive.

Another important quantity is the inflation risk premium, ϵ, defined by

$$\epsilon = R - r + \mathbb{E}\left[\frac{dP^{-1}}{P^{-1}}\right]$$

[5]In addition to the utility function parameters.

When investing in a nominal money market account, instead of a real one, the investor takes a risk, due to inflation uncertainty, which is compensated by the inflation risk premium. From Eqs.(31), (32) and (12), one obtains, after rearrangement, the following expression :

$$\begin{aligned}\epsilon = {} & \frac{\alpha-\beta}{\alpha}\rho - \mu_\alpha - \frac{1-\alpha}{\alpha}\mu_M + \frac{1-\alpha^2}{\alpha^2}\sigma_M^2 - \frac{1}{\alpha(1-\alpha)}\sigma_\alpha\sigma_M(\alpha^2+\alpha-2) + \sigma_\eta^2 \\ & - \frac{2-\alpha}{1-\alpha}\sigma_\alpha\sigma_\eta - \frac{2-\alpha}{\alpha}\sigma_M\sigma_\eta + \frac{1}{1-\alpha}\sigma_\alpha^2 \\ & + \lambda\mathbb{E}\Big[\frac{1}{1+\gamma_M}\Big(\gamma_M + \frac{\alpha}{1-\alpha}\gamma_\alpha - 1 - \frac{1}{1+\gamma_\eta}\big(\frac{1}{\alpha}\gamma_M - \gamma_\eta\gamma_\alpha + 1\big)\Big) \\ & - \frac{\alpha\gamma_\alpha(\gamma_M+1) + (1-\alpha(1+\gamma_\alpha))(\gamma_M - \alpha\gamma_\eta)}{(1-\alpha)\gamma_M + \alpha(1-\alpha)(1+\gamma_\eta)}\Big]\end{aligned} \tag{35}$$

We focus our analysis on the jump component. We notice that the additional term, with respect to those obtained in the pure diffusion case by LP, has monetary parameters. This indicates that the equilibrium value of the inflation risk premium is still affected by monetary policy in the presence of jumps.

1.2 The pricing kernels

In this subsection, we derive, in equilibrium, the dynamics of the real and nominal pricing kernels under the jump-diffusion framework. The pricing kernel approach is an alternative to the martingale method since it delivers the time t price of a claim delivering a certain payoff at a given maturity. The real pricing kernel is defined to be the investor's marginal utility of consumption (U_c) which is equal, in our specific case, to $e^{-\rho t}\beta\frac{1}{c}$. Using Eq.(21) with strict equality as well as the specific form of the indirect utility function, we obtain the expression of the optimal consumption in terms of real wealth :

$$c = \rho\beta w \tag{36}$$

The real pricing kernel is therefore :

$$\xi(t) = \frac{1}{\rho}e^{-\rho t}\frac{1}{w(t)} \tag{37}$$

Applying Ito's lemma to Eq.(37) yields

$$\frac{d\xi(t)}{\xi(t-)} = -rdt - \lambda\mathbb{E}\big[\gamma_\xi\big]dt + \sigma_\xi dZ(t) + \gamma_\xi dN(t) \tag{38}$$

where $\sigma_\xi = -\sigma_w$ and $\gamma_\xi = -\dfrac{\gamma_w}{1+\gamma_w}$. One can notice that the market price of jump volatility risk, γ_ξ is affected by monetary parameters which means that money non-neutrality is preserved under the jump-diffusion extension.
Once the real pricing kernel is known, we can deduce the nominal one $\zeta(t)$ through the following relation :

$$\zeta(t) = \xi(t)P^{-1}(t) \tag{39}$$

Applying Ito's lemma to Eq.(39) gives :

$$\frac{d\zeta(t)}{\zeta(t-)} = -Rdt - \lambda \mathbb{E}\big[\gamma_\zeta\big]dt + \sigma_\zeta dZ(t) + \gamma_\zeta dN(t) \tag{40}$$

where $\sigma_\zeta = -\sigma_M - \dfrac{\alpha}{1-\alpha}\sigma_\alpha$ and $\gamma_\zeta = -\dfrac{\gamma_M}{1+\gamma_M} - \dfrac{\alpha}{1-\alpha}\dfrac{\gamma_\alpha}{1+\gamma_M}$ The jump volatility risk of the nominal pricing kernel depends, as expected, on monetary parameters. As we will show below, the investment to real wealth ratio terms depend, in fact, on the monetary state variable and monetary policy parameters. This is due to the log-separable assumption of the utility function.

2 A specialized economy

In order to derive the dynamics of the nominal interest rate further specification of the dynamics of the state variables as well as those of the production and money supply are needed. We consider, in the following, the case of two state variables, one real and one nominal which dynamics are given by

$$dY_\eta(t) = \mu_{Y_\eta} Y_\eta dt + Y_\eta \sigma'_{Y_\eta} dZ(t) + Y_\eta \gamma'_{Y_\eta} dN(t) \tag{41}$$

$$dY_M(t) = \mu_{Y_M} Y_M dt + Y_M \sigma'_{Y_M} dZ(t) + Y_M \gamma'_{Y_M} dN(t) \tag{42}$$

respectively. μ_{Y_η} and μ_{Y_M} are positive constants, σ_{Y_η}, σ_{Y_M}, γ_{Y_η} and γ_{Y_M} are $1 \times (N+2)$ vectors of constants, $Z(t)$ is an $1 \times (N+2)$ dimensional Wiener process, and $N(t)$ is an $1 \times (N+2)$ dimensional Poisson process.

Furthermore, Eqs.(2) and(3) are assumed to have the following forms

$$\frac{d\eta(t)}{\eta(t-)} = \mu_\eta Y_\eta(t)dt + \sqrt{Y_\eta(t)}\sigma'_\eta dZ(t) + Y_\eta(t)\gamma'_\eta dN(t) \tag{43}$$

$$\frac{dM(t)}{M(t-)} = \mu_M Y_M(t)dt + \sqrt{Y_M(t)}\sigma'_M dZ(t) + Y_M(t)\gamma'_M dN(t) \tag{44}$$

respectively. μ_η and μ_M are positive constants, and σ_η, σ_M, γ_η are $1 \times (N+2)$ vectors of constants. While σ_M and σ_η are positive, γ_η and γ_M can take negative values.

2.1 A nominal interest rate model

To determine the dynamics of the nominal short term interest rate $R(t)$, we rely, first, on a standard result[6], that is the nominal interest rate is equal, in equilibrium, to a ratio of marginal rates of substitution between real money balances and consumption :

$$R = \frac{U_m}{U_c} \tag{45}$$

which gives in our separable utility function case

$$R = \frac{1-\beta}{1-\alpha}\rho \tag{46}$$

The presence of a monetary state variable implies a stochastic behavior of the investment to wealth ratio α. This stems from the fact, as it will be shown below, that α can be written as a function of Y_M. Therefore, α will be constant if we no longer assume a monetary state variable in our framework. This feature results in a constant nominal interest rate which is, of course, a non-realistic assumption.
First, using Eqs.(46) and (32), we straightforwardly prove that

$$\beta\rho + \mu_M Y_M - \sigma_M^2 Y_M + \lambda\mathbb{E}\big[\frac{\gamma_M Y_M}{1+\gamma_M Y_M}\big] = \\ \alpha\Big(\rho - \mu_\alpha + \mu_M Y_M - \sigma_M^2 Y_M + \sigma_\alpha\sigma_M Y_M + \lambda\mathbb{E}\big[\frac{\gamma_M Y_M}{1+\gamma_M Y_M} - \frac{\gamma_\alpha}{1+\gamma_M Y_M}\big]\Big) \tag{47}$$

[6]See proposition 5 in Lioui and Poncet (2004).

which can be rewritten in a compact form as

$$\alpha = f(Y_M) \tag{48}$$

Applying Ito's lemma to $f(Y_M)$ yields the SDE that governs changes in α :

$$\frac{d\alpha}{\alpha} = \underbrace{\Big(\frac{f'}{f}\mu_{Y_M}Y_M + \frac{1}{2}\frac{f''}{f}\sigma^2_{Y_M}Y_M^2\Big)}_{=\mu_\alpha} dt + \underbrace{\frac{f'}{f}\sigma_{Y_M}Y_M^2}_{=\sigma_\alpha} dZ + \underbrace{\Big(\frac{f(Y_M(1+\gamma))}{f} - 1\Big)}_{=\gamma_\alpha} dN \tag{49}$$

Substituting the expressions of μ_α, σ_α and γ_α in Eq.(47) and rearranging gives

$$\begin{aligned}\frac{\beta\rho + \mu_M Y_M - \sigma_M^2 Y_M + \lambda\mathbb{E}\big[1 - \frac{f(Y_M(1+\gamma))}{1+\gamma_M Y_M}\big]}{\rho + \mu_M Y_M - \sigma_M^2 Y_M + \lambda\mathbb{E}\big[\frac{\gamma_M Y_M}{1+\gamma_M Y_M}\big]} = f - f'\frac{\mu_{Y_M} - \sigma_{Y_M}\sigma_M Y_M^{3/2}}{\rho + \mu_M Y_M - \sigma_M^2 Y_M^2 + \lambda\mathbb{E}\big[\frac{\gamma_M Y_M}{1+\gamma_M Y_M}\big]} \\ - \frac{1}{2}f''\frac{\sigma^2_{Y_M}Y_M^2}{\rho + \mu_M Y_M - \sigma_M^2 Y_M + \lambda\mathbb{E}\big[\frac{\gamma_M Y_M}{1+\gamma_M Y_M}\big]}\end{aligned} \tag{50}$$

One can notice that, due to the log separable utility function, we assumed, the investment to real wealth ratio depends solely on the nominal state variable. Therefore, in the case of state variable occurring only at the real level, the expression of α reduces down to

$$\alpha = \frac{\beta\rho + \mu_M - \sigma_M^2 + \lambda\mathbb{E}\big[\frac{\gamma_M}{1+\gamma_M}\big]}{\rho + \mu_M - \sigma_M^2 + \lambda\mathbb{E}\big[\frac{\gamma_M}{1+\gamma_M}\big]} \tag{51}$$

leading to a constant α which is only affected by monetary quantities.
Solving for f (Eqs.(48) and (50)) is not, in general, possible since the coefficients depend themselves on the monetary state variable. Nevertheless, following LP, α can be approximated as follows :

$$\alpha \approx \frac{\beta\rho + \mu_M Y_M - \sigma_M^2 Y_M + \kappa_1}{\rho + \mu_M Y_M - \sigma_M^2 Y_M + \kappa_2} \tag{52}$$

where $\kappa_1 = \lambda\mathbb{E}\big[1 - \dfrac{f(Y_M(1+\gamma))}{1+\gamma_M Y_M}\big]$ and $\kappa_2 = \lambda\mathbb{E}\big[\frac{\gamma_M Y_M}{1+\gamma_M Y_M}\big]$. A further approximation can performed by assuming that $\kappa_1 \approx \kappa_2$. Both approximations can be tested by considering the special case where the economy is only affected by a real state variable which implies that Eq.(52) becomes (51).

Using Eq.(52) and applying Ito's lemma to Eq.(46) yields, under the jump-diffusion setting, the dynamics of the nominal short interest rate :

Proposition 4 *In the special economy, the nominal interest rate evolves, under the jump-diffusion framework, according to the following SDE :*

$$dR(t) = (R(t) - (\rho + \kappa_2))\bar{\mu}dt + (R(t) - (\rho + \kappa_2))\bar{\sigma}dz(t) + R(t)\gamma_R dn(t) \tag{53}$$

where we have introduced one-dimensional Wiener process, $dz(t)$, and Poisson process, $dn(t)$. Also, we define $\bar{\mu} \equiv \mu_{Y_M}$, $\bar{\sigma} \equiv \sigma_{Y_M}$ and $R(t)\gamma_R \equiv R(Y_M(1 + \gamma_{Y_M}) - R(Y_M)$ where γ_R is chosen so that the nominal interest rate remains positive.

In this special case, the short-term nominal interest rate process is an extension of a lognormal model to a jump-diffusion process. Other models of the short-term interest rate that has lognormal distribution are developed by Black et al. (1990) and Black and Karasinski (1991). These models, however, have been obtained in an arbitrage-free framework.

3 Conclusion

This paper extends to a jump-diffusion setting the monetary equilibrium of Lioui and Poncet (2004). Assuming a particular dynamics of the state variable as well as a log-separable utility function, we obtain a model of the short-term nominal interest rate that follows a mixed process of diffusion and jumps. Empirical evidence in the literature shows that introducing jumps is a desired feature to capture the dynamics of an interest rate. Moreover, we determined the inflation risk premium and derived the dynamics of the general price level. A future line of research may focus on estimating the model's parameters from market data on treasury bills and bonds. In addition, relying on the jump-diffusion dynamics of the inflation rate we can price various contingent claims from the fast growing inflation-linked derivatives market, hence, providing an extension of Lioui and Poncet (2005) where pricing of inflation-indexed options has been achieved in the pure diffusion case.

A Appendix

Proof. Eq.(38)

Given the expression of the real pricing kernel (Eq. (37)), we apply Ito's lemma and obtain

$$\frac{d\xi(t)}{\xi(t)} = -(\rho + \mu_w - \sigma_w^2)dt - \sigma_w dW(t) - \frac{\gamma_w}{1+\gamma_w}dN(t) \tag{A-1}$$

where

$$\mu_w = \mu_\eta - \frac{c}{\alpha w} - \mu_\alpha - \frac{1-\alpha}{\alpha}\mu_M + \sigma'_\alpha\sigma_\alpha - \frac{1-\alpha}{\alpha}\sigma'_M\sigma_\eta + \frac{2-\alpha}{\alpha}\sigma'_M\sigma_\alpha + \frac{1-\alpha}{\alpha^2}\sigma'_M\sigma_M - \sigma'_\alpha\sigma_\eta \tag{A-2}$$

with, at equilibrium, $c = \rho w\beta$. Hence,

$$\mu_w = \mu_\eta - \frac{\rho\beta}{\alpha} - \mu_\alpha - \frac{1-\alpha}{\alpha}\mu_M + \sigma'_\alpha\sigma_\alpha - \frac{1-\alpha}{\alpha}\sigma'_M\sigma_\eta + \frac{2-\alpha}{\alpha}\sigma'_M\sigma_\alpha + \frac{1-\alpha}{\alpha^2}\sigma'_M\sigma_M - \sigma'_\alpha\sigma_\eta \tag{A-3}$$

in addition,

$$\sigma_w = \sigma_\eta - \sigma_\alpha - \frac{1-\alpha}{\alpha}\sigma_M \tag{A-4}$$

which becomes

$$\sigma_w^2 = \sigma_\eta^2 - 2\sigma_\eta\sigma_\alpha + \sigma_\alpha^2 + \left(\frac{1-\alpha}{\alpha}\right)^2\sigma_M^2 - 2\frac{1-\alpha}{\alpha}\sigma_\eta\sigma_M + 2\frac{1-\alpha}{\alpha}\sigma_\alpha\sigma_M \tag{A-5}$$

Therefore,

$$\begin{aligned}\rho + \mu_w - \sigma_w^2 = \rho + \mu_\eta - \frac{\rho\beta}{\alpha} - \mu_\alpha - \frac{1-\alpha}{\alpha}\mu_M + \sigma'_\alpha\sigma_\alpha - \\ \frac{1-\alpha}{\alpha}\sigma'_M\sigma_\eta + \frac{2-\alpha}{\alpha}\sigma'_M\sigma_\alpha + \frac{1-\alpha}{\alpha^2}\sigma'_M\sigma_M - \sigma'_\alpha\sigma_\eta \\ - \sigma_\eta^2 + 2\sigma_\eta\sigma_\alpha - \sigma_\alpha^2 - \left(\frac{1-\alpha}{\alpha}\right)^2\sigma_M^2 + 2\frac{1-\alpha}{\alpha}\sigma_\eta\sigma_M - 2\frac{1-\alpha}{\alpha}\sigma_\alpha\sigma_M\end{aligned} \tag{A-6}$$

Using Eqs.(32) and (46), we obtain

$$\rho = \frac{\beta}{\alpha}\rho + \mu_\alpha + \frac{1-\alpha}{\alpha}\mu_M - \frac{1-\alpha}{\alpha}\sigma_M^2 - \sigma_\alpha\sigma_M + \frac{1-\alpha}{\alpha}\lambda\mathbb{E}\left[\frac{\gamma_M}{1+\gamma_M} + \frac{\alpha}{1-\alpha}\frac{\gamma_\alpha}{1+\gamma_M}\right] \tag{A-7}$$

Thus,

$$\rho + \mu_w - \sigma_w^2 = \mu_\eta - \sigma_\eta^2 + \sigma_\eta\sigma_\alpha + \frac{1-\alpha}{\alpha}\sigma_M\sigma_\eta + \frac{1-\alpha}{\alpha}\lambda\mathbb{E}\left[\frac{\gamma_M}{1+\gamma_M} + \frac{\alpha}{1-\alpha}\frac{\gamma_\alpha}{1+\gamma_M}\right] \tag{A-8}$$

Given Eq. (31), Eq.(A-8) becomes :

$$\rho+\mu_w-\sigma_w^2 = r-\lambda\mathbb{E}\left[\frac{\gamma_\eta\gamma_M}{\alpha(1+\gamma_\eta)(1+\gamma_M)}+\frac{1}{1+\gamma_M}+\frac{\gamma_\eta\gamma_\alpha-1}{(1+\gamma_\eta)(1+\gamma_M)}-\frac{1-\alpha}{\alpha}\frac{\gamma_M}{1+\gamma_M}-\frac{\gamma_\alpha}{1+\gamma_M}\right] \tag{A-9}$$

We notice that terms under the expectation can be rewritten as Or l'expression sous $\frac{\gamma_w}{1+\gamma_w}$.
Thus,

$$\rho + \mu_w - \sigma_w^2 = r - \lambda\mathbb{E}\left[\frac{\gamma_w}{1+\gamma_w}\right] \tag{A-10}$$

which completes the proof. □

Proof. Eq.(40) Applying Ito's lemma to Eq. (39) gives,

$$d\xi P^{-1} = P^{-1}\xi\mu_\xi dt + P^{-1}\xi\sigma_\xi dW(t) + P^{-1}\xi\gamma_\xi + P^{-1}\xi(-\mu_P + \sigma_P^2)dt - P^{-1}\xi\sigma_P dW(t) - P^{-1}\xi\frac{\gamma_P}{1+\gamma_P}dN(t) \tag{A-11}$$

$$= P^{-1}\xi\left\{(\mu_\xi - \mu_P + \sigma_P^2 - \sigma_P\sigma_\xi)dt + (\sigma_\xi - \sigma_P)dW(t) + \left(\gamma_\xi - \frac{\gamma_P}{1+\gamma_P} - \frac{\gamma_P}{1+\gamma_P}\gamma_\xi\right)dN(t)\right\} \tag{A-12}$$

First the diffusion term :

$$\sigma_\xi - \sigma_P = -\sigma_w - \sigma_P \tag{A-13}$$

using Eqs.(14) and (18) obtains :

$$\sigma_\zeta = -\frac{\alpha}{1-\alpha}\sigma_\alpha - \sigma_M \tag{A-14}$$

The drift term :

$$-r + \lambda \mathbb{E}\big[\frac{\gamma_w}{1+\gamma_w}\big] - \mu_p + \sigma_p^2 + \sigma_p \sigma_w = \tag{A-15}$$

$$- \mu_\eta + \sigma_\eta^2 - \sigma_\alpha \sigma_\eta - \frac{1-\alpha}{\alpha}\sigma_M \sigma_\eta - \lambda \mathbb{E}\big[\frac{\gamma_\eta \gamma_M}{\alpha(1+\gamma_\eta)(1+\gamma_M)} + \frac{1}{1+\gamma_M} + \frac{\gamma_\eta \gamma_\alpha - 1}{(1+\gamma_\eta)(1+\gamma_M)}\big] + \lambda \mathbb{E}\big[\frac{\gamma_w}{1+\gamma_w}\big] \tag{A-16}$$

$$- \mu_p + \sigma_p^2 + \sigma_p \sigma_w \tag{A-17}$$

rearranging :

$$= -\lambda \mathbb{E}\big[\frac{1-\alpha}{\alpha}\frac{\gamma_M}{1+\gamma_M} + \frac{\gamma_\alpha}{1+\gamma_M}\big] + \underbrace{\frac{1}{1-\alpha}\sigma_\alpha \sigma_M + \frac{1}{\alpha}\sigma_M^2 - \frac{1}{1-\alpha}\mu_\alpha - \frac{1}{1-\alpha}\mu_M}_{=\frac{1}{\alpha}\lambda \mathbb{E}\left[\frac{\gamma_M}{1+\gamma_M} + \frac{\alpha}{1-\alpha}\frac{\gamma_\alpha}{1+\gamma_M}\right] - \frac{R}{\alpha} + \frac{\rho}{\alpha}} - \frac{\beta\rho}{\alpha} \tag{A-18}$$

Simplifying,

$$\mu_\zeta = -R - \lambda \mathbb{E}\left[\frac{-\gamma_M}{1+\gamma_M} - \frac{\alpha}{1-\alpha}\frac{\gamma_\alpha}{1+\gamma_M}\right] \tag{A-19}$$

□

Proof. Proposition (4)

from Eq. (52), we have :

$$1-\alpha = \frac{\rho(1-\beta)}{\rho + \mu_M Y_M - \sigma_M^2 Y_M + \kappa_2} \tag{A-20}$$

which gives :

$$\rho + \mu_M Y_M - \sigma_M^2 Y_M + \kappa_2 = \frac{\rho(1-\beta)}{1-\alpha} = R \tag{A-21}$$

The expression of the nominal short interest rate, R, is :

$$R(Y_M) = \rho + \kappa_2 + (\mu_M - \sigma_M^2) Y_M \tag{A-22}$$

Applying Ito's lemma :

$$dR = (\mu_M - \sigma_M^2) dY_M + [R(Y_M(1+\gamma_{Y_M})) - R(Y_M)] dN(t) \tag{A-23}$$

$$= (\mu_M - \sigma_M^2)\mu_{Y_M} Y_M dt + (\mu_M - \sigma_M^2)\sigma_{Y_M} Y_M dZ(t) + [R(Y_M(1+\gamma_{Y_M})) - R(Y_M)] dN(t) \tag{A-24}$$

Using $R - \rho - \kappa_2 = (\mu_M - \sigma_M^2)Y_M$ completes the proof. $\square$

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