Necessary and Sufficient Conditions
for No Static Arbitrage among European Calls

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Abstract

Under the assumption of the absence of arbitrage, European call prices on a given asset must satisfy well-known inequalities, which have been described in the landmark paper Merton (1973). If we further assume that there is no interest rate volatility and that the underlying pays continuously deterministic dividends, cross maturity inequalities must also be satisfied by the call prices.

We show that there exists an arbitrage free model, which is consistent with the call prices, if these inequalities are satisfied. Furthermore, we describe an algorithm to obtain a realistic calibrated martingale Markov chain model, using the notions of entropy and of copula.

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AMS classification codes: 60J10, 60J20, 60J22.
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1 Introduction

In trying to understand the dynamics of a given underlying, the financial markets only tell part of the story. For example, options quotes yield some information about the marginal distributions at specific future times. However, a complete specification of the possible future behavior of the underlying remains largely undetermined.

Consequently, to bet on an arbitrage strategy based on a particular market structure can be very risky as explained in the enlightening binomial-tree based example of Carr, Geman, Madan and Yor (2003) (henceforth CGMY). It makes more sense to restrict oneself to static strategies in which the future behavior of the underlying is less relevant. Nevertheless, it is not sufficient to make the notion of arbitrage independent of the real “statistical measure”.

Indeed, classical static arbitrage strategies involving vertical (or call) spreads, butterfly spreads and calendar spreads do not depend on the ”statistical measure”. But is it possible to claim that the elimination of these strategies is sufficient to prevent static arbitrage without specifying a ”statistical measure”? The answer is no. Indeed, if the price of a call is zero and the probability that the underlying will be greater than the strike at maturity is positive, then to buy the call is an arbitrage strategy. Likewise, if the price of a call is positive and the probability that the underlying will be greater than or equal to the strike at maturity is zero, then to sell the call is an arbitrage strategy.

In this paper, we will show that these necessary conditions for the absence of static arbitrage are also sufficient for the existence of a market model which is consistent with the call quotes and further, is (static) arbitrage free. We generalize the results of Carr and Madan (2004) by allowing the underlying to be a deterministic dividend paying stock, the short interest rate to be a deterministic function of time, and the grid of the available strikes to be finite and not rectangular. Furthermore, we describe an algorithm to construct a more realistic calibrated martingale Markov chain model.

The structure of this paper is given as follows. Section 2 describes the assumptions we make and introduces some calendar versions of the classical vertical and butterfly spreads; Section 3 describes a way to construct a marginal distribution consistent with call quotes corresponding to a given maturity; Section 4 uses this construction to exhibit a martingale measure, which prevents arbitrage; Section 5 recalls briefly why the conditions are necessary for the absence of static arbitrage; Section 6 explains how to construct a more realistic calibrated Markov chain model; Section 7 explains how to relax some of the starting assumptions; Section 8 concludes.

2 Assumptions and definitions

We assume that we have at our disposal a finite set of call quotes on a given non-dividend paying stock. The maturities, indexed in increasing order, will be denoted by \( (T_i)_{1 \leq i \leq m} \). For a given maturity \( T_i \), \( n_i \) quotes are available corresponding to the strikes \( K_{ij} < \ldots < K_{ij}^n \) and will be denoted by \( (C_{ij})_{1 \leq j \leq n_i} \). We also assume that there is no interest rate. For each maturity, we will add a quote corresponding to a call struck at 0. By convention, \( K_{i0} = 0 \) and \( C_{i0} = S_0 \) - the initial stock price - by arbitrage.

Now we will extend the classical definitions of vertical, calendar and butterfly spreads because the grid of available strikes is not rectangular. The calendar version of a vertical (resp. butterfly) spread must be thought of as a linear combination of calendar spreads and a classical vertical (resp. butterfly) spread.

**Definition 2.1 (Vertical Spreads).** \( \forall i \in [1, m], \)
\[
VS_i^j = \frac{C_{i,j-1}^j - C_{i,j}^j}{K_j^i - K_{j-1}^i}, \quad 1 \leq j \leq n_i \\
VS_i^0 = 1
\]

**Definition 2.2 (Calendar (Vertical) Spreads).** \( \forall i_1, i_2 \in [1, m] \ s.t. \ i_1 \leq i_2, \ \forall j_1 \in [0, n_{i_1}], \forall j_2 \in [0, n_{i_2}] \) s.t. \( K_{j_1}^{i_1} \geq K_{j_2}^{i_2} \)
\[
CVS_{i_1,i_2}^{j_1,j_2} = C_{j_2}^{i_2} - C_{j_1}^{i_1}
\]
Definition 2.3 ((Calendar) Butterfly Spreads). \( \forall i, i_1, i_2 \in [1, m] \text{ s.t. } i_1 \leq i_1 \text{ and } i \leq i_2, \forall j \in [0, n_1], \forall j_1 \in [0, n_1], j_2 \in [0, n_2] \text{ s.t. } K_{j_1} < K_j < K_{j_2} \)

\[
CB_S^{i_1, i_2} = \frac{CV_{j_1, j_2}^{i_1, i_2}}{K_j - K_{j_1} - K_{j_2}}
\]

3 Construction of discrete risk-neutral marginal distributions

Lemma 3.1 (Discrete Distribution). If \( N \in \mathbb{N}^+ \), \( k_0 = 0 < \cdots < k_j < \cdots < k_N \) are \( N + 1 \) real numbers and \( P_C : \mathbb{R}^+ \rightarrow \mathbb{R} \) is a function, which is continuous, convex, linear and decreasing on each interval \([k_j, k_{j+1}]\) with \( 0 \leq j \leq N - 1 \) (with a slope greater than or equal to -1 on \([k_0, k_1]\)) and zero after \( k_N \) then the following distribution

\[
\mu = \sum_{j=0}^{N} q_j \delta_{k_j}
\]

with

\[
q_0 = 1 - \frac{P_C(k_0) - P_C(k_1)}{k_1 - k_0}
\]

\[
q_j = \frac{P_C(k_{j-1}) - P_C(k_j)}{k_j - k_{j-1}} - \frac{P_C(k_j) - P_C(k_{j+1})}{k_{j+1} - k_j}, \ 1 \leq j \leq N - 1
\]

\[
q_N = \frac{P_C(k_{N-1})}{k_N - k_{N-1}}
\]

is such that:

\[
g(K) = \int_0^\infty (x - K)^+ d\mu = P_C(K) \tag{1}
\]

for all \( K \in \mathbb{R}^+ \).

Proof. The non-negativity of \( q_0 \) is ensured by the condition on the slope of \( P_C \) on \([k_0, k_1]\). Further \( q_j \geq 0 \) (1 \leq j \leq N - 1) due to the convexity of \( P_C \). Finally \( q_N \) is non-negative because \( P_C \) is also non-negative. Moreover,

\[
\sum_{k=0}^{N} q_k = q_0 + \frac{P_C(k_0) - P_C(k_1)}{k_1 - k_0} - \frac{P_C(k_{N-1}) - P_C(k_N)}{k_N - k_{N-1}} + q_N
\]

\[
= 1 - \frac{P_C(k_0) - P_C(k_1)}{k_1 - k_0} + \frac{P_C(k_0) - P_C(k_1)}{k_1 - k_0} - \frac{P_C(k_{N-1}) - P_C(k_N)}{k_N - k_{N-1}} + \frac{P_C(k_{N-1})}{k_N - k_{N-1}}
\]

\[
= 1 + \frac{P_C(k_{N-1})}{k_N - k_{N-1}}
\]

\[
= 1
\]

We now show that the distribution results in a linear call price function \( g \) between two nodes \([k_j, k_{j+1}]\), 0 \leq j \leq N - 1. \( \forall K \in (k_j, k_{j+1}] \), we have:

\[
g(K) = \sum_{k=0}^{N} q_k (k_k - K)^+
\]

\[
= \sum_{j+1 \leq k \leq N} q_k (k_k - K)
\]

\[
= \left( \sum_{j+1 \leq k \leq N} q_k k_k \right) - K \left( \sum_{j+1 \leq k \leq N} q_k \right)
\]
Moreover $g$ is also right-continuous in $k_j$. Indeed,

$$g(K) \xrightarrow{K \to k_j^+} \sum_{j+1 \leq k \leq N} q_k(k_k - k_j) = \sum_{j \leq k \leq N} q_k(k_k - k_j) = g(k_j)$$

Clearly $g$ is zero on $[k_N, +\infty)$. Now if we prove that the slopes are equal on each segment $[k_j; k_{j+1}]$, $0 \leq j \leq N - 1$, we will be able to conclude that the functions $g$ and $P_C$ are equal. Accordingly,

$$g(k_{j+1}) - g(k_j) = \sum_{j+1 \leq k \leq N} q_k(k_k - k_{j+1}) - \sum_{j+1 \leq k \leq N} q_k(k_k - k_j)$$

$$= (k_j - k_{j+1}) \sum_{j+1 \leq k \leq N} q_k$$

$$= (k_j - k_{j+1}) \frac{P_C(k_j) - P_C(k_{j+1})}{k_{j+1} - k_j}$$

$$= P_C(k_{j+1}) - P_C(k_j)$$

In the next section, we will use this lemma to construct marginal distributions. The quoted strikes will be among the $k_j$ and their corresponding prices will be given by the function $P_C$. Consequently the distribution $\mu$ will be a risk-neutral marginal distribution for the given maturity.

4 Sufficient conditions for the existence of a martingale measure

4.1 Outline of the proof

If we are able to construct marginal distributions, that are consistent with the call quotes and the initial stock price, non-decreasing in the convex order (henceforth NDCO) and have the same mean, then we can conclude that there exists a martingale consistent with all these marginal distributions due to Kellerer theorem.
Theorem 4.1 (Kellerer theorem (1972)). Let \((\mu_t)_{t \in [0,T]}\) be a family of probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) with first moment, such that for \(s < t\) \(\mu_t\) dominates \(\mu_s\) in the convex order, i.e. for each convex function \(\phi : \mathbb{R} \to \mathbb{R}\) \(\mu_t\)-integrable for each \(t \in [0,T]\), we must have:

\[
\int_{\mathbb{R}} \phi d\mu_t \geq \int_{\mathbb{R}} \phi d\mu_s
\]

Then there exists a Markov process \((M_t)_{t \in [0,T]}\) with these marginals for which it is a submartingale. Furthermore if the means are independent of \(t\) then \((M_t)_{t \in [0,T]}\) is a martingale.

Proof. See Kellerer (1972).

4.2 Construction of marginal distributions that are NDCO

Proposition 4.2. Let us assume that:

- \(\forall i \in [1, m],\)
  \[
  C^i_j \geq 0, \quad 0 \leq j \leq n_i \\
  VS^i_j \in [0, 1] \quad 1 \leq j \leq n_i \\
  VS^i_j > 0 \quad \text{if } 1 \leq j \leq n_i \text{ and } C^i_{j-1} > 0
  \]

- \(\forall i_1, i_2 \in [1, m] \text{ s.t. } i_1 < i_2, \forall j_1 \in [0, n_{i_1}], \forall j_2 \in [0, n_{i_2}]\)
  \[
  CVS_{i_1, i_2}^i \geq 0 \quad \text{if } K_{j_1}^{i_1} \geq K_{j_2}^{i_2} \\
  CVS_{i_1, i_2}^i > 0 \quad \text{if } K_{j_1}^{i_1} > K_{j_2}^{i_2} \text{ and } C_j^{i_1} > 0
  \]

- \(\forall i, i_1, i_2 \in [1, m] \text{ s.t. } i \leq i_1 \text{ and } i \leq i_2, \forall j \in [0, n_i], \forall j_1 \in [0, n_{i_1}], \forall j_2 \in [0, n_{i_2}] \text{ s.t. } K_{j_1}^{i_1} < K_j^i < K_{j_2}^{i_2}\)
  \[
  CBS_{i_1, i_2}^{i_1, i_2} \geq 0
  \]

then there exists \(m\) risk-neutral distributions \((\phi_j)_{1 \leq j \leq m}\) corresponding to the different maturities. Let us define \(\phi_0 \equiv \delta_{S_0}\). \((\phi_j)_{0 \leq j \leq m}\) are NDCO and their means are all equal to \(S_0\).

Before proving this proposition, we would like to give some insight on the required conditions. If a martingale is consistent with the quoted call prices, then the general call price function would be non-increasing and convex in the strike, as well as non-decreasing in the maturity. For example, we can easily conclude that a situation as the one described in Table 1 is not compatible with the existence of a pricing martingale. Indeed if there was a pricing martingale, then \(x \geq 30\) and \(x \leq 20\), which is impossible.

<table>
<thead>
<tr>
<th>Maturity \ Strike</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>x</td>
<td>30</td>
</tr>
<tr>
<td>3 months</td>
<td>20</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 1

Another more complicated example is given by Table 2.

<table>
<thead>
<tr>
<th>Maturity \ Strike</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>—</td>
<td>37</td>
<td>—</td>
</tr>
<tr>
<td>3 months</td>
<td>(x_1)</td>
<td>(x_2)</td>
<td>30</td>
</tr>
<tr>
<td>6 months</td>
<td>40</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 2
Here the condition on the calendar butterfly spread is violated and this can be explained as follows: if there was a pricing martingale, then we would have:

\[ 40 \geq x_1 \]  
(The price is non-decreasing with maturity.)
\[ \frac{x_1 + 30}{2} \geq x_2 \]  
(The price is convex with respect to the strike.)
\[ x_2 \geq 37 \]  
(The price is non-decreasing with maturity.)

and this would result in \( 35 = \frac{40 + 30}{2} \geq 37 \). This last example allows us to understand why the strike corresponding to the nearest maturity is always between the two other strikes in the definition of the calendar butterfly spread. The same kind of conclusions could not be drawn in a situation such as the one described in Table 3.

<table>
<thead>
<tr>
<th>Maturity \ Strike</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
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<tbody>
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<td>1 month</td>
<td>40</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>3 months</td>
<td>—</td>
<td>—</td>
<td>30</td>
</tr>
<tr>
<td>6 months</td>
<td>—</td>
<td>37</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 3

Proof. Lemma 3.1 tells us how to associate in a one-to-one way a distribution with a \( T_i \)-call price function if the latter respects the given conditions. Consequently, to construct the marginal distributions, we will construct call price functions which have the right properties and which go through the call quotes at \( T_i \). Finally we will consider the corresponding distributions as candidates.

The distributions must be NDCO. Notably the corresponding call price functions must be non-decreasing with maturity. By only considering the quotes at a given maturity, it is possible that the corresponding constructed distributions will not possess this characteristic (See Figure 2). Nevertheless, observe that if we add the points corresponding to quotes of longer maturities to the quotes of maturity \( T_i \), the frontier of their convex hull\(^1\) seems to have all of the features we are looking for.

\[ \text{Figure 2: Frontier at } T_i. \]

To ensure the monotonicity of the distributions, we must also properly specify the call price function to the right of the greatest quoted strike - if the corresponding price is not zero. Let us determine the maximum of the non-zero slopes between two points, which could be consecutive on the frontier a priori.

\(^1\)In the definition of the convex hull we should restrict the slopes of the linear functions to be non-positive.
We adopt the following notation:

\[
\text{MaxSlope} \equiv \max \left( \left\{ \frac{CV S_{i,j}^{i,i+1}}{K_{j_1}^{i_1} - K_{j_2}^{i_2}} \right\}_{1 \leq i_1, i_2 \leq m, 0 \leq j_1 \leq n_{i_1}, 0 \leq j_2 \leq n_{i_2}} \right) \cap \mathbb{R}^+
\]

If \( C_{i_3}^{i_2} > 0 \), we will add a point \((K_{n_{i_3}+1}^{i_3}, 0)\) such that the slope between \((K_{n_{i_3}+1}^{i_3}, C_{n_{i_3}}^{i_3})\) and \((K_{n_{i_3}+1}^{i_3}, 0)\) is equal to \(\text{MaxSlope}/2\). By convention, if \( C_{n_i}^{i} = 0 \) then \( K_{n_{i+1}}^{i+1} = K_{n_i}^{i} + 1 \).

\[
A_j^{i} \equiv (K_j^{i}, C_j^{i})
\]

\[
K_{n_i+1}^{i} = K_{n_i}^{i} - 2C_{n_i}^{i}/\text{MaxSlope}
\]

\[
A_i^{i+1} = (K_{n_i}^{i+1}, 0)
\]

\[
A_i^{i} = \bigcup_{j=0}^{n_{i+1}} A_j^{i}
\]

\[
B_i^{i} = \text{Convex hull of } \bigcup_{j=i}^{m} A_j^{i}
\]

We will show that the frontier of \(B_i^{i}\) corresponds to a risk-neutral distribution for the calls of maturity \(T_i\) and furthermore that these distributions are NDCO.

It is obvious that \(Fr(B_i^{i})\), the frontier of \(B_i^{i}\), is continuous, piece-wise linear, convex and that the nodes are among the \(K_k^{i}\) (with corresponding value \(C_k^{i}\), \(i \leq k \leq m, 0 \leq j \leq n_k + 1\)).

Let us show that the slope of the frontier on the first segment is greater than or equal to -1. The value of the frontier in \(K_0^{i} = 0\) is clearly \(C_0^{i} = S_0\). Let us denote by \(K_1^{i}\) the smallest non-zero node of the frontier.

\[
\frac{C_j^{i} - C_0^{i}}{K_j^{i}} = \frac{C_j^{i} - C_0^{i}}{K_1^{i}}
\]

\[
= \frac{1}{K_1^{i}} \sum_{k=1}^{j_1} V S_k^{i} (K_k^{i} - K_{k-1}^{i})
\]

\[
\geq \frac{-V S_0^{i}}{K_1^{i}} \sum_{k=1}^{j_1} (K_k^{i} - K_{k-1}^{i}) \quad (V S_j^{i} \geq V S_{j+1}^{i}, 0 \leq j < n_{i_1} \text{ by convexity})
\]

\[
\geq -\frac{K_1^{i_1} - K_1^{i_0}}{K_1^{i}}
\]

\[
\geq -1
\]

We now show that the frontier is zero after a given value. Since \((K_{n_{i_3}+1}^{i_3}, 0)\) is a point in the convex hull \(B_i^{i}\) and the linear functions involved in the definition of the convex hull have non-positive slopes, the frontier is zero on \([K_{n_{i_3}+1}^{i_3}, 0]\). Denote by \(K_2^{i}\) the smallest strike which corresponds to a node of the frontier whose call value is zero.

Let us examine the monotonicity of \(Fr(B_i^{i})\). We consider two consecutive nodes of the frontier \(K_{j_1}^{i_1}\) and \(K_{j_2}^{i_2}\) with \(K_{j_1}^{i_1} < K_{j_2}^{i_2}\). Since we know that the frontier is flat at the right of \(K_2^{i}\), we only need to prove that the segment \([A_{j_1}^{i_1}; A_{j_2}^{i_2}]\) is decreasing if \(K_{j_2}^{i_2} \leq K_2^{i}\).

- If \(C_{j_2}^{i_2} = 0\), then by definition of \(K_2^{i}\), we have \(K_{j_2}^{i_2} = K_2^{i}\). Because \(K_{j_1}^{i_1} < K_{j_2}^{i_2} = K_2^{i}\), \(C_{j_1}^{i_1} > 0\), i.e. the slope is negative.

- If \(C_{j_2}^{i_2} > 0\) and \(i_2 \leq i_1\), then the condition on the (calendar) vertical spreads shows the monotonicity. \((i_1 \neq n_{i_1} + 1 \text{ and } i_2 \neq n_{i_2} + 1 \text{ since the two call prices are positive})\).

- If \(C_{j_2}^{i_2} > 0\) and \(i_2 > i_1\), \([A_{j_1}^{i_1}; A_{j_2}^{i_2}]\) is below the segment \([A_{j_1}^{i_1}; A_{j_1+1}^{i_1}]\) because \(A_{j_2}^{i_2}\) is on the frontier. (Remember that \(i_1 \neq n_{i_1} + 1\)). Since \([A_{j_1}^{i_1}; A_{j_1+1}^{i_1}]\) is decreasing (because of the condition on the vertical spreads if \(j_1 < n_{i_1}\) and the definition of \(\text{MaxSlope}\) if \(j_1 = n_{i_1}\)), the segment \([A_{j_1}^{i_1}; A_{j_2}^{i_2}]\) must be decreasing as well.
We now show that the frontier goes through the points $A^j_i$ for $0 \leq j \leq n_i$. For $j = 0$, it is obvious since all the $A^0_i$ correspond to the same point. Let us consider a point $A^j_i$ that does not belong to the frontier $(1 \leq j \leq n_i)$. Denote by $K^i_{j_1}$ (resp. $K^i_{j_2}$) the greatest (resp. smallest) node of the frontier which is less (resp. greater) than $K^i_j$. ($K^i_{j_2}$ exists since $K^i_{n_i+1}$ is on the frontier). Recall that $i \leq i_1$ and $i \leq i_2$.

- If $j_1 \leq n_i$ and $j_2 \leq n_i$, then the non-negativity of the (calendar) butterfly spreads contradicts the fact that $A^j_i$ does not belong to the frontier.

- If $j_1 \leq n_i$ and $j_2 = n_i + 1$, then we have the following two cases:
  - If $C^i_{n_2} = 0$, then $A^i_{n_2}$ is also on the frontier. Since $A^i_{n_1}$ and $A^i_{n_2+1}$ are consecutive on the frontier, $C^i_{j_1} = 0$. The condition on the (calendar) vertical spreads between the points $A^i_{j_1}$ and $A^i_{j_2}$ imposes that $C^i_j \leq 0$. Combined with the non-negativity condition on call quotes, this ensures us that the point $A^j_i$ lies in the flat frontier, which is a contradiction.
  - If $C^i_{n_2} > 0$, then the slope of the segment $[A^i_{j_1}; A^i_{j_2+1}]$ must be less than the slope of the segment $[A^i_{n_1}; A^i_{n_2+1}]$, since $A^i_{j_1}$ and $A^i_{j_2}$ are two consecutive nodes on the frontier. Its slope is therefore greater than MaxSlope/2. But the slope of the segment $[A^i_{j_1}; A^i_{j_2}]$ is greater than the slope of $[A^i_{n_1}; A^i_{n_2+1}]$, since $A^i_j$ does not belong to the frontier. Consequently, the slope of $[A^i_{j_1}; A^i_{j_2}]$ is greater than MaxSlope/2. We can exclude the case where $C^i_j = 0$, since in this case $A^j_i$ always belong to the frontier. If $C^i_j > 0$, then the slope of $[A^i_{j_1}; A^i_j]$ is strictly negative because of the condition on the (calendar) vertical spread and must therefore be less than or equal to MaxSlope, which is a contradiction.

- If $j_1 = n_i + 1$, then $C^i_{j_2} = 0$ because the corresponding point belongs to the frontier, which is flat at the right of $K^i_{n_i+1}$. We have the following two cases:
  - If $C^i_{n_i+1} = 0$, the condition on the (calendar) vertical spread between the points $A^i_{j_1}$ and $A^i_{j_2}$ imposes that $C^i_j \leq 0$. Combined with the non-negativity condition on call quotes, this ensures that the point $A^j_i$ lies in the flat frontier, which is a contradiction.
  - If $C^i_{n_i+1} > 0$, then the slope of the segment $[A^i_{n_1}; A^i_{j_2}]$ is greater than the slope of $[A^i_{n_1}; A^i_{n_2+1}]$, which is given by MaxSlope/2. But this slope is strictly negative because of the condition on the vertical (calendar) spread ($C^i_j > 0$ since $A^i_j$ does not belong to the frontier). Consequently, it must be less than or equal to MaxSlope. This is a contradiction.

We have proved that $Fr(B^i)$ satisfies all the conditions of Lemma 3.1. Therefore we can associate to it a distribution $\mu_i$ satisfying (1). Moreover this distribution is risk-neutral for the calls of maturity $T_i$ since $Fr(B^i)$ goes through the points $A^i_j$. We still have to prove that these distributions are NDZCO and that their mean is constant over time.

For a given strike, the prices are non-decreasing with the maturity since $B^{i+1} \subseteq B^i$ implies that $Fr(B^i) \subseteq Fr(B^{i+1}), 1 \leq i < m$. The call price function corresponding to the distribution $\phi_0 \equiv \delta_{S_0}$ starts at $S_0$, has a slope of -1 until $S_0$ and is zero after. Consequently, it is less than the one corresponding to $\phi_1$, which starts at $S_0$, has a starting slope greater than or equal to -1 and is convex. The prices of European put options are also non-decreasing with maturity because of put-call parity. Since any convex function can be approximated by linear combinations with positive weights of put and call functions, the distributions are NDZCO.

Finally the means of the different distributions are constant over time because all the calls struck at 0 have the same price.

\[\square\]

5 Necessity of the conditions

The conditions of Proposition 4.2 are of course necessary for the existence of a pricing martingale but they are also necessary for the absence of static arbitrage.
First of all, let us be precise in what we mean by static strategies. At the initial time, one is allowed to take long and short positions in the options and in the stock. Furthermore, as explained in CGMY (2003), one has the possibility to short the stock for a given future period of time if the stock is greater than a specified value at the beginning of the period. Under the assumption of no static arbitrage and of a frictionless market, it is costless. In other words, at the initial time, one can purchase at zero cost, a security whose payoff at time $T_2 (> T_1 > 0)$ is:

$$1_{\{S_{T_1} > K\}}(S_{T_1} - S_{T_2})$$

The inequalities of Proposition 4.2 are classic necessary conditions and will not be detailed in this paper. We just consider, as an example, the case of a (vertical) calendar spread constructed from the call of strike $K_1$, maturity $T_1$ and price $C_1$, and the call of strike $K_2$, maturity $T_2$ and price $C_2$ ($K_1 > K_2$ and $T_1 < T_2$). The strategy is to sell the call of maturity $T_1$, buy the call of maturity $T_2$ and short one share if the stock at $T_1$ is greater than $K_1$. The price of this strategy is of course $C_2 - C_1$ and its payoff at $T_2$ is:

$$-(S_{T_1} - K_1)^+ + 1_{\{S_{T_1} > K_1\}}(S_{T_1} - S_{T_2}) + (S_{T_2} - K_2)^+ \geq 1_{\{S_{T_1} > K_1\}}(K_1 - K_2)$$

We remark that the payoff of this strategy is always non-negative. Consequently, its price should also be non-negative, i.e. $C_1 \leq C_2$. Moreover, if the probability that $S_{T_1} > K_1$ is positive and $K_1 > K_2$, then the payoff is non-negative and positive with positive probability. Therefore, in this case, the price of this strategy should be positive and we notice that the condition $P(S_{T_1} > K_1) > 0$ is equivalent to $C_1 > 0$, since we assumed no static arbitrage.

## 6 Construction of a more intuitive and realistic model

The previous construction of the pricing martingale offers little intuition and may result in an unrealistic model for the stock. In this section, we will construct a martingale Markov chain model, which will be calibrated to the quoted call prices. First we will describe an algorithm based on entropy maximization to construct more realistic, NDCO, marginal distributions. Once these distributions are constructed, we will be able to claim the existence of martingale transition matrices due to the Sherman-Stein-Blackwell theorem, as explained for example in Davis (2004). To choose realistic transition matrices, we will once again describe an algorithm based on the notion of entropy and of Brownian copula.

### 6.1 Construction of marginal distributions

The marginal distributions we have constructed may be quite unrealistic. Indeed, the number of possible states for the stock may be decreasing with maturity. We will describe here, an algorithm allowing to construct more realistic discrete marginal distributions.

Let $(k_i)_{0 \leq i \leq N}$ be a finite subset of $\mathbb{R}^+$ containing all the $(K^i_j)_{1 \leq i \leq m, 0 \leq j \leq n, i+1}$. This set will be the support of the marginal distributions.

In the proof of Proposition 4.2, we have already constructed some marginal distributions having the required features and $(k_i)_{0 \leq i \leq N}$ as a support. To choose a more realistic set of marginal distributions, we will use the notion of entropy, which is appealing for several reasons. One reason is that the less information a discrete distribution contains, the higher the entropy - the maximum being reached for a uniform distribution. A direct consequence is that the entropy maximization will distribute the masses among all the $(k_i)$ in a more uniform manner.

Let us denote by $\mathcal{P}$ the set of the call price functions satisfying the assumptions of Lemma 3.1 and by $\mathcal{Q}$ the set of the corresponding distributions, whose support is consequently $(k_i)$. Let us denote by $P^1_{C_{\text{old}}}$ the call price function corresponding to the maturity $T_1$ constructed in the proof of Proposition 4.2. The new call price function, $P^1_{C_{\text{new}}}$, corresponding to the maturity $T_1$, is obtained by maximizing the entropy$^2$:

$$\max_{q_C \in \mathcal{Q}} \left( - \sum_{i=0}^{N} q_C(k_i) \log(q_C(k_i)) \right)$$

$^2$By convention $0 \times \log(0) = 0$ in the definition of entropy.
s.t. $0 \leq P_C \leq P^1_C$ and $\forall j \in [0, n_1], P_C(K^1_j) = P^1_C(K^1_j)$

Now, let us assume that we have constructed the new marginal distributions until the maturity $T_{i-1}$. $P_{C,new}^i$ is obtained by maximizing:

$$\max_{q_C \in Q} \left( -\sum_{i=0}^{N} q_C(k_i) \log(q_C(k_i)) \right)$$

s.t. $P_{C,new}^{i-1} \leq P_C \leq P^i_C$ and $\forall j \in [0, n_i], P_C(K^i_j) = P^i_C(K^i_j)$

We remark that the new distributions are risk neutral since:

$$\forall i \in [1, m], \forall j \in [0, n_i], P_{C,new}^i(K^i_j) = P^i_C(K^i_j) = C^i_j$$

Moreover these distributions are NDCO by construction. A numerical example is given in Appendix 1.

### 6.2 Construction of martingale transition matrices

Now that we have constructed risk neutral marginal distributions, which are NDCO, we will focus on the construction of martingale transition matrices. The Sherman-Stein-Blackwell theorem ensures the existence of such matrices. This theorem is a generalization to the N-dimensional case of the Hardy-Littlewood-Polya theorem (1929) and has been generalized by Strassen (1965) to probability measures in $\mathbb{R}^N$. The Kellerer theorem (1972), which we stated in section 4.1 is a continuous time version of the latter.

**Theorem 6.1 (Sherman-Stein-Blackwell theorem).** If $X = (a_1, ..., a_n)$ is a finite set in $\mathbb{R}^N$, and $q^1$ and $q^2$ are probability measures on $X$ such that

$$\sum_{i=1}^{n} \phi(a_i)q^1(a_i) \leq \sum_{i=1}^{n} \phi(a_i)q^2(a_i)$$

for every continuous convex function $\phi$ defined on the convex hull of $X$, then there is a martingale transition matrix $(q_{i,j})_{1 \leq i, j \leq n}$ such that:

$$q_{i,j} \geq 0 \quad \text{for } 1 \leq i, j \leq n$$

$$\sum_{i=1}^{n} q_{i,j} = 1 \quad \text{for } 1 \leq j \leq n$$

$$\sum_{i=1}^{n} q_{i,j}q^1(a_i) = q^2(a_j) \quad \text{for } 1 \leq j \leq n$$

$$\sum_{i=1}^{n} q_{i,j}a_j = a_i \quad \text{for } 1 \leq i \leq n$$

**Proof.** See Davis (2004) for Strassen’s proof restricted to the finite case.

Now that the existence of the transition matrices is established, we would like to have at our disposal an extra criterion to choose realistic ones. Once again we will minimize under constraints the Kullback-Leibler distance of the joint distributions:

$$D(J^i | J^{i,prior}) = \sum_{1 \leq \alpha, \beta \leq N} J^i_{\alpha, \beta} \log \left( \frac{J^i_{\alpha, \beta}}{J^{i,prior}_{\alpha, \beta}} \right)$$
(where \( J_{α,β}^i \equiv P(S_{T_i} = k_α, S_{T_{i+1}} = k_β) \) is the joint distribution of \( (S_{T_i}, S_{T_{i+1}}) \) and \( (J_{α,β}^{i,prior})_{1 \leq α, β \leq N} \) is a given prior joint distribution), under the following constraints:

\[
J_{α,β}^i ≥ 0
\]

\[
\sum_{1 ≤ β ≤ N} J_{α,β}^i = q^i(α) \quad 1 ≤ α ≤ N \text{ (Law of total probability)}
\]

\[
\sum_{1 ≤ α ≤ N} J_{α,β}^i = q^{i+1}(β) \quad 1 ≤ β ≤ N \text{ (Chapman-Kolmogorov equations)}
\]

\[
\sum_{1 ≤ β ≤ N} k_β J_{α,β}^i = q^i(α)k_α \quad 1 ≤ α ≤ N \text{ (Martingale conditions)}
\]

where the \( T_i \)-marginal distribution obtained in Section 6.1 is denoted by \( (q^i(α))_{1 ≤ α ≤ N} \).

As explained in Cover and Thomas (1991) or in Avellaneda et al (2000), this problem can be reduced to the unconstrained maximization problem in \( (λ, μ, ν) \) of:

\[
W(λ, μ, ν) = -\log \left( \sum_{1 ≤ α, β ≤ N} J_{α,β}^{i, prior} e^{λ_α + ν_α k_β + µ_β} \right) + \sum_{1 ≤ α ≤ N} q^i(α)(λ_α + ν_α k_α) + \sum_{1 ≤ β ≤ N} µ_β q^{i+1}(β)
\]

Indeed, if \( (λ^0, μ^0, ν^0) \) is a critical point of \( W \), the joint distribution defined by:

\[
J_{α,β}^{i, prior} = \frac{J_{α,β}^{i, prior}}{\sum_{1 ≤ α, β ≤ N} J_{α,β}^{i, prior} e^{λ^0_α + ν^0_α k_β + µ^0_β}}
\]

is a solution of the initial problem, provided that \( J_{α,β}^{i, prior} ≥ 0 \).

We now need to specify a prior joint distribution. One way is to discretize the Brownian copula as explained in Carr and Cousot (2004).

Recall that once the marginal distributions are specified, the knowledge of the copula of \( (S_{T_i}, S_{T_{i+1}}) \), \( C_{S_{T_i}, S_{T_{i+1}}} \), and of the joint distribution are equivalent (See Darsow, Nguyen and Olsen (1992)). Indeed, under the assumption of bilinear interpolation, the copula is uniquely determined by its values at the points \( (\sum_{a=1}^α q^i(a), \sum_{b=1}^β q^{i+1}(b)) \):

\[
C_{α,β}^i = C_{S_{T_i}, S_{T_{i+1}}} \left( \sum_{a=1}^α q^i(a), \sum_{b=1}^β q^{i+1}(b) \right) = \sum_{a=1}^α \sum_{b=1}^β J_{α,b}^i
\]

Conversely, if the copula is given, the joint distribution is uniquely specified by:

\[
J_{α,β}^i = C_{α,β}^i - C_{α-1,β}^i - C_{α,β-1}^i + C_{α-1,β-1}^i
\]

for \( 1 ≤ α, β ≤ N \).

We would like the joint distribution to imply a realistic copula. Consequently, one way is to choose as a prior joint distribution the one implied by the Brownian copula characterized in Darsow et al (1992):

\[
C_{B_{T_i}, B_{T_{i+1}}}(x, y) = \int_0^x \Phi \left( \frac{\sqrt{T_{i+1}}Φ^{-1}(y) - \sqrt{T_i}Φ^{-1}(u)}{\sqrt{T_{i+1}} - T_i} \right) du
\]

where \( Φ \) is the cumulative normal distribution.

\[
C_{B_{α,β}}^i = C_{B_{T_i}, B_{T_{i+1}}} \left( \sum_{a=1}^α q^i(a), \sum_{b=1}^β q^{i+1}(b) \right)
\]

\[
J_{α,β}^{i, prior} = C_{B_{α,β}}^i - C_{α-1,β}^i - C_{α,β-1}^i + C_{α-1,β-1}^i
\]

We remark that this so defined joint distribution \( (J_{α,β}^{i, prior}) \) has the right marginal distributions, but unfortunately does not infer a martingale. That is why we need to minimize the Kullback-Leibler distance. A numerical example is given in Appendix 2.
7 Generalization to the case of a deterministic dividend paying stock in presence of deterministic interest rates

In this section we generalize the results of Section 4 by assuming that the stock pays continuously a deterministic dividend $q$ and that the short interest rate is a deterministic function $r$. By convention, we will add a call struck at 0 for each maturity $T_i$: $K_0^i = 0$ and $C_0^i = S_0 e^{-\int_0^{T_i} q_u^i du}$ with $S_0$ the initial stock price.

**Proposition 7.1.** If the following conditions are fulfilled

- $\forall i \in [1, m], \forall j \in [0, n_i]$  
  \[ C_j^i \geq 0 \]

- $\forall i \in [1, m], \forall j_1, j_2$ s.t. $0 \leq j_1 < j_2 \leq n_i$  
  \[ \frac{C_{j_1}^i - C_{j_2}^i}{K_{j_2}^i - K_{j_1}^i} \in [0, e^{-\int_0^{T_i} r_u du}] \]

- $\forall i \in [1, m], \forall j_1, j_2$ s.t. $0 \leq j_1 < j_2 \leq n_i$  
  \[ \frac{C_{j_1}^i - C_{j_2}^i}{K_{j_2}^i - K_{j_1}^i} > 0 \text{ if } C_{j_1}^i > 0 \]

- $\forall i, i_1, i_2 \in [1, m]$ s.t. $i \leq i_1$ and $i \leq i_2$, $\forall j \in [0, n_i], \forall j_1 \in [0, n_{i_1}], \forall j_2 \in [0, n_{i_2}]$ s.t. $e^{-\int_0^{T_i} (r_u - q_u) du} K_{j_1}^i \leq e^{-\int_0^{T_i} (r_u - q_u) du} K_{j_2}^i < e^{-\int_0^{T_i} (r_u - q_u) du} K_{j_2}^i$  
  \[ \frac{e^{-\int_0^{T_i} q_u du} C_{j_1}^i - e^{-\int_0^{T_i} q_u du} C_{j_2}^i}{e^{-\int_0^{T_i} (r_u - q_u) du} K_{j_1}^i - e^{-\int_0^{T_i} (r_u - q_u) du} K_{j_2}^i} \geq 0 \]

then there exists a process $(S_t)_{t \geq 0}$ starting at the initial stock price, such that $(S_t e^{-\int_0^{T_i} (r_u - q_u) du})_{t \geq 0}$ is a martingale and further $\forall i$ s.t. $1 \leq i \leq m$, $\forall j$ s.t. $0 \leq j \leq n_i$  

\[ C_j^i = \mathbb{E}[e^{-\int_0^{T_i} r_u du} (S_t - K_j^i)^+] \]

**Proof.** We will use the results we have proven in the case of a non-dividend paying stock where the interest rates are null. First let us define $\forall i$ s.t. $1 \leq i \leq m$, $\forall j$ s.t. $0 \leq j \leq n_i$  

\[ K_j^i \equiv K_j^i e^{-\int_0^{T_i} (r_u - q_u) du} \]

\[ C_j^i \equiv e^{\int_0^{T_i} q_u du} C_j^i \]

We can see that $K_j^i$ and $C_j^i$ satisfy the hypothesis of Theorem 4.2. Consequently, there exists a continuous time Markov martingale $(M_t)$ such that:

\[ C_j^i = \mathbb{E}[(M_t - K_j^i)^+] \]

This can be written as:

\[ C_j^i = \mathbb{E}[e^{-\int_0^{T_i} r_u du} (M_t e^{\int_0^{T_i} (r_u - q_u) du} - K_j^i)^+] \]

Let us define the process $(S_t)_{t \geq 0}$ as follows: $S_t \equiv M_t e^{\int_0^{T_i} (r_u - q_u) du}$. The last property we have to verify is that this process starts at $S_0$. The two processes have the same starting value by definition and the process $(M_t)_{t \geq 0}$, which is a martingale, starts at $C_0^i = e^{\int_0^{T_i} q_u du} C_0^i = S_0$.  \[ \square \]
The inequalities of Proposition 7.1 are again necessary for the absence of static arbitrage. Furthermore, the algorithm described in Section 6 to obtain a realistic Markov chain model can also be generalized easily. Indeed, it is possible to return to the no interest rate volatility and no dividend case by the following one-to-one transformation,

\[
\begin{align*}
\mathcal{K}_{ij}^t & \equiv \mathcal{K}_{ij}^t e^{-\int_0^{T_i} (r_u - q_u) du} \\
\mathcal{C}_{ij}^t & \equiv e^{\int_0^{T_i} q_u du} \mathcal{C}_{ij}^t
\end{align*}
\]

Then we can apply the algorithm previously described, and finally return to the general case by the inverse of the above transformation.

8 Conclusion

The elimination of the well-known static strategies involving European calls are in general not sufficient to prevent static arbitrage: the limitation to static strategies is not sufficient to make the notion of arbitrage independent of the underlying "statistical measure". Nevertheless, we have proved that these necessary conditions are sufficient for the existence of an arbitrage free model consistent with the call quotes. We also focused on providing an algorithm explaining how to construct a realistic, arbitrage free, Markov chain model. We used the notion of entropy as well as the notion of Brownian copula to choose realistic marginal and joint distributions. An application of this model is for example the pricing of mildly path-dependent claims, (e.g. globally floored, locally capped, compounding cliquet options), as explained in Carr and Cousot (2004).
References


Appendix 1: Construction of marginal distributions, a numerical example

Let us consider the following situation: the current price of the non dividend paying stock is $S_0 = 100$ and the available call quotes are detailed in Table 1. (We assume that there are no interest rates.)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Strike</th>
<th>Implied Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>70</td>
<td>28.0169 %</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>25.1770 %</td>
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<tr>
<td></td>
<td>90</td>
<td>22.2610 %</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>19.2448%</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>16.6954 %</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>15.8362 %</td>
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<tr>
<td></td>
<td>130</td>
<td>15.9489 %</td>
</tr>
<tr>
<td>3m</td>
<td>70</td>
<td>27.0431 %</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>21.5104 %</td>
</tr>
<tr>
<td></td>
<td>95</td>
<td>20.1062 %</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>18.7155 %</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>17.4310 %</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>16.4006 %</td>
</tr>
<tr>
<td></td>
<td>130</td>
<td>15.4949 %</td>
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<td>150</td>
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<td>90</td>
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<tr>
<td></td>
<td>100</td>
<td>18.1727 %</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>16.2882 %</td>
</tr>
<tr>
<td></td>
<td>140</td>
<td>15.0552 %</td>
</tr>
<tr>
<td></td>
<td>180</td>
<td>16.1231 %</td>
</tr>
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<td>1y</td>
<td>50</td>
<td>30.6808 %</td>
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<td></td>
<td>90</td>
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<tr>
<td></td>
<td>100</td>
<td>18.1727 %</td>
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<td>15.0552 %</td>
</tr>
<tr>
<td></td>
<td>180</td>
<td>16.1231 %</td>
</tr>
</tbody>
</table>

Table 4: Quotes, which satisfy the hypothesis of Proposition 4.2.

Instead of setting the slope of the segment $[A_{n_1}^i, A_{n_1+1}^i]$ to $MaxSlope/2$, we could have set it to $\alpha \times MaxSlope$ with $0 < \alpha < 1$ without changing the validity of the proof of Proposition 4.2. Here we choose $\alpha = 0.999$ to reduce the value of $K_{n_1+1}^i$.

\[
\alpha = 0.999 \\
MaxSlope = 10^{-5} \\
n_1 = 7 \\
n_2 = 8 \\
n_3 = 6 \\
K_{n_1+1}^1 = 251.21212121 \\
K_{n_2+1}^2 = 160.10101010 \\
K_{n_3+1}^3 = 250.70707071
\]

The $(k_i)$ in this example are the minimal set of all the $(K_j^i)_{1 \leq j \leq m, 1 \leq i \leq n_1+1}$. In the following tables, the approximations up to $10^{-8}$ of the distributions $q_i^C$ and $q_i^C,\text{new}$ are given.
Table 5: the distributions $q_C^1$ and $q_{C, new}^1$.

<table>
<thead>
<tr>
<th>$k_j$</th>
<th>$q_C^1(k_j)$</th>
<th>$q_{C, new}^1(k_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2.9000 \times 10^{-4}$</td>
<td>$9 \times 10^{-8}$</td>
</tr>
<tr>
<td>50</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.01468 \times 10^{-3}$</td>
</tr>
<tr>
<td>70</td>
<td>$1.487000 \times 10^{-2}$</td>
<td>$1.414523 \times 10^{-2}$</td>
</tr>
<tr>
<td>80</td>
<td>$6.485000 \times 10^{-2}$</td>
<td>$6.485000 \times 10^{-2}$</td>
</tr>
<tr>
<td>90</td>
<td>$2.0652000 \times 10^{-1}$</td>
<td>$1.2573229 \times 10^{-1}$</td>
</tr>
<tr>
<td>95</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.6157543 \times 10^{-1}$</td>
</tr>
<tr>
<td>100</td>
<td>$3.8499000 \times 10^{-4}$</td>
<td>$2.0761345 \times 10^{-4}$</td>
</tr>
<tr>
<td>105</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.9317767 \times 10^{-4}$</td>
</tr>
<tr>
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<td>$2.7638000 \times 10^{-4}$</td>
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</tr>
<tr>
<td>120</td>
<td>$4.9070000 \times 10^{-2}$</td>
<td>$4.9070000 \times 10^{-2}$</td>
</tr>
<tr>
<td>130</td>
<td>$2.975000 \times 10^{-3}$</td>
<td>$2.91428 \times 10^{-3}$</td>
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<tr>
<td>140</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.1154 \times 10^{-4}$</td>
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<tr>
<td>150</td>
<td>$4.510 \times 10^{-5}$</td>
<td>$4.08 \times 10^{-6}$</td>
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<td>$K_{n+1}^2$</td>
<td>$9.90 \times 10^{-6}$</td>
<td>$1.0 \times 10^{-1}$</td>
</tr>
<tr>
<td>180</td>
<td>$0 \times 10^{-8}$</td>
<td>$0 \times 10^{-8}$</td>
</tr>
<tr>
<td>$K_{n+1}^3$</td>
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<td>$0 \times 10^{-8}$</td>
</tr>
<tr>
<td>$K_{n+1}^4$</td>
<td>$0 \times 10^{-8}$</td>
<td>$0 \times 10^{-8}$</td>
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</tbody>
</table>

Figure 3: Marginal distributions at 3 month maturity, $q_C^1$ (left) and $q_{C, new}^1$ (right).
<table>
<thead>
<tr>
<th>$k_j$</th>
<th>$q_{2C}^C(k_j)$</th>
<th>$q_{2C,new}^C(k_j)$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>$1.77200 \times 10^{-3}$</td>
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<td>$3.43800 \times 10^{-3}$</td>
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<td>$8.934500 \times 10^{-2}$</td>
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<td>$0 \times 10^{-8}$</td>
<td>$7.691490 \times 10^{-2}$</td>
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<td>90</td>
<td>$1.6476500 \times 10^{-4}$</td>
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<td>120</td>
<td>$0 \times 10^{-8}$</td>
<td>$7.462215 \times 10^{-2}$</td>
</tr>
<tr>
<td>130</td>
<td>$6.681500 \times 10^{-2}$</td>
<td>$2.808188 \times 10^{-2}$</td>
</tr>
<tr>
<td>140</td>
<td>$0 \times 10^{-8}$</td>
<td>$2.84410 \times 10^{-3}$</td>
</tr>
<tr>
<td>150</td>
<td>$1.71010 \times 10^{-4}$</td>
<td>$2.8805 \times 10^{-3}$</td>
</tr>
<tr>
<td>$K_{n+1}$</td>
<td>$9.90 \times 10^{-6}$</td>
<td>$9.90 \times 10^{-6}$</td>
</tr>
<tr>
<td>180</td>
<td>$0 \times 10^{-8}$</td>
<td>$0 \times 10^{-8}$</td>
</tr>
<tr>
<td>$K_{n+1}^2$</td>
<td>$0 \times 10^{-8}$</td>
<td>$0 \times 10^{-8}$</td>
</tr>
<tr>
<td>$K_{n+1}^2$</td>
<td>$0 \times 10^{-8}$</td>
<td>$0 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 6: the distributions $q_{2C}^C$ and $q_{2C,new}^C$.

Figure 4: Marginal distributions at 6 month maturity, $q_{2C}^C$ (left), $q_{2C,new}^C$ (right).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$k_j$ & $q_3^C(k_j)$ & $q_{3C,new}(k_j)$ \\
\hline
0 & $1.77200 \times 10^{-3}$ & $1.77200 \times 10^{-3}$ \\
50 & $8.958550 \times 10^{-2}$ & $3.669584 \times 10^{-2}$ \\
70 & $0 \times 10^{-3}$ & $6.375830 \times 10^{-2}$ \\
80 & $0 \times 10^{-8}$ & $8.404205 \times 10^{-2}$ \\
90 & $2.5834250 \times 10^{-1}$ & $1.1077879 \times 10^{-1}$ \\
95 & $0 \times 10^{-8}$ & $1.0530604 \times 10^{-1}$ \\
100 & $2.2209600 \times 10^{-4}$ & $1.0010365 \times 10^{-4}$ \\
105 & $0 \times 10^{-8}$ & $1.3640667 \times 10^{-1}$ \\
110 & $3.3337433 \times 10^{-1}$ & $1.8587513 \times 10^{-1}$ \\
120 & $0 \times 10^{-8}$ & $9.514572 \times 10^{-2}$ \\
130 & $0 \times 10^{-8}$ & $4.870317 \times 10^{-2}$ \\
140 & $9.364917 \times 10^{-2}$ & $2.493017 \times 10^{-2}$ \\
150 & $0 \times 10^{-8}$ & $5.30849 \times 10^{-3}$ \\
$K_{n+1}^2$ & $0 \times 10^{-8}$ & $1.11284 \times 10^{-3}$ \\
180 & $1.93760 \times 10^{-3}$ & $5.125 \times 10^{-5}$ \\
$K_{n+1}^3$ & $9.90 \times 10^{-6}$ & $9.90 \times 10^{-6}$ \\
$K_{n+1}^4$ & $0 \times 10^{-8}$ & $0 \times 10^{-8}$ \\
\hline
\end{tabular}
\caption{the distributions $q_3^C$ and $q_{3C,new}$.}
\end{table}

Figure 5: Marginal distributions at 1 year maturity, $q_3^C$ (left), $q_{3C,new}$ (right).
We also tried with a thiner grid. The next distributions have been obtained with:

\[ (k_i) = \{0, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100, 105, \]
\[ 110, 115, 120, 130, 135, 140, 145, 150, K_{n+1}^2, K_{n+1}^3, K_{n+1}^4 \} \]

![Table 8: the distributions \( q_C \) and \( q_{C,new} \).](image)

![Figure 6: Marginal distributions at 3 month maturity, \( q_C \) (left) and \( q_{C,new} \) (right).](image)
<table>
<thead>
<tr>
<th>$k_j$</th>
<th>$q_C^2(k_j)$</th>
<th>$q_{C,new}^2(k_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.77200 × 10^{-3}</td>
<td>2.59 × 10^{-6}</td>
</tr>
<tr>
<td>30</td>
<td>0 × 10^{-8}</td>
<td>8.372 × 10^{-8}</td>
</tr>
<tr>
<td>35</td>
<td>0 × 10^{-8}</td>
<td>1.6621 × 10^{-4}</td>
</tr>
<tr>
<td>40</td>
<td>0 × 10^{-8}</td>
<td>3.2501 × 10^{-4}</td>
</tr>
<tr>
<td>45</td>
<td>0 × 10^{-8}</td>
<td>6.3851 × 10^{-4}</td>
</tr>
<tr>
<td>50</td>
<td>3.43800 × 10^{-3}</td>
<td>1.25458 × 10^{-3}</td>
</tr>
<tr>
<td>55</td>
<td>0 × 10^{-8}</td>
<td>2.46148 × 10^{-3}</td>
</tr>
<tr>
<td>60</td>
<td>0 × 10^{-8}</td>
<td>4.83131 × 10^{-3}</td>
</tr>
<tr>
<td>65</td>
<td>0 × 10^{-8}</td>
<td>9.48246 × 10^{-3}</td>
</tr>
<tr>
<td>70</td>
<td>8.934500 × 10^{-2}</td>
<td>1.860712 × 10^{-2}</td>
</tr>
<tr>
<td>75</td>
<td>0 × 10^{-8}</td>
<td>2.769184 × 10^{-2}</td>
</tr>
<tr>
<td>80</td>
<td>0 × 10^{-8}</td>
<td>4.120652 × 10^{-2}</td>
</tr>
<tr>
<td>85</td>
<td>0 × 10^{-8}</td>
<td>6.131944 × 10^{-2}</td>
</tr>
<tr>
<td>90</td>
<td>1.6476500 × 10^{-1}</td>
<td>9.124919 × 10^{-2}</td>
</tr>
<tr>
<td>95</td>
<td>1.1972000 × 10^{-1}</td>
<td>1.1972000 × 10^{-1}</td>
</tr>
</tbody>
</table>

Table 9: the distributions $q_C^2$ and $q_{C,new}^2$.

Figure 7: Marginal distributions at 6 month maturity, $q_C^2$ (left), $q_{C,new}^2$ (right).
<table>
<thead>
<tr>
<th>$k_j$</th>
<th>$q^3_{C}(k_j)$</th>
<th>$q^3_{C,new}(k_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.77200 \times 10^{-3}$</td>
<td>$3.196 \times 10^{-5}$</td>
</tr>
<tr>
<td>30</td>
<td>$0 \times 10^{-8}$</td>
<td>$8.5533 \times 10^{-4}$</td>
</tr>
<tr>
<td>35</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.47926 \times 10^{-3}$</td>
</tr>
<tr>
<td>40</td>
<td>$0 \times 10^{-8}$</td>
<td>$2.55834 \times 10^{-3}$</td>
</tr>
<tr>
<td>45</td>
<td>$0 \times 10^{-8}$</td>
<td>$4.42459 \times 10^{-3}$</td>
</tr>
<tr>
<td>50</td>
<td>$8.958550 \times 10^{-4}$</td>
<td>$7.65221 \times 10^{-4}$</td>
</tr>
<tr>
<td>55</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.026585 \times 10^{-2}$</td>
</tr>
<tr>
<td>60</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.377218 \times 10^{-2}$</td>
</tr>
<tr>
<td>65</td>
<td>$0 \times 10^{-8}$</td>
<td>$1.847611 \times 10^{-2}$</td>
</tr>
<tr>
<td>70</td>
<td>$0 \times 10^{-8}$</td>
<td>$2.478669 \times 10^{-2}$</td>
</tr>
<tr>
<td>75</td>
<td>$0 \times 10^{-8}$</td>
<td>$3.325264 \times 10^{-2}$</td>
</tr>
<tr>
<td>80</td>
<td>$0 \times 10^{-8}$</td>
<td>$4.461017 \times 10^{-2}$</td>
</tr>
<tr>
<td>85</td>
<td>$0 \times 10^{-8}$</td>
<td>$5.984692 \times 10^{-2}$</td>
</tr>
<tr>
<td>90</td>
<td>$2.5834250 \times 10^{-1}$</td>
<td>$8.028781 \times 10^{-2}$</td>
</tr>
<tr>
<td>95</td>
<td>$0 \times 10^{-8}$</td>
<td>$9.479990 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 10: the distributions $q^3_{C}$ and $q^3_{C,new}$.

Figure 8: Marginal distributions at 1 year maturity, $q^3_{C}$ (left), $q^3_{C,new}$ (right).
Appendix 2: Construction of a joint distribution, a numerical example

If the \((k_i)\) are the minimal set of all the \((K^j)^{1\leq i \leq m, 1\leq j \leq n_i+1}\), we obtain:

![Figure 9: Joint distribution of \((S_{T_1}, S_{T_2})\).](image)

![Figure 10: The corresponding transition matrix.](image)
The following joint distribution and the corresponding transition matrix have been obtained with:

\( (k_i) = \{0, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100, 105, 110, 115, 120, 130, 135, 140, 145, 150, K_{n_2+1}^2, 180, K_{n_3+1}^3, K_{n_2+1}^2 \} \)

Figure 11: Joint distribution of \((S_{T_1}, S_{T_2})\).

Figure 12: The corresponding transition matrix.