Value at Risk Using Skew Elliptical Distributions

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The Value-at-Risk concept relies on the fundamental returns normality hypothesis. Given that this does not hold empirically, many authors relaxed it allowing for fat tails and leptokurtosis. But the pitfall is that they don’t take care of asymmetry. Therefore, the aim of this paper is to try to fit stock returns with both the Skew Normal and the Skew t densities which belong to a class of skew elliptical distributions. Our objective is to compute more realistic Value-at-Risk estimates. After, presenting the VaR concept, come the skew elliptical densities. Finally, our results on the CAC40 Index returns, are consistent with stylised facts.

Keywords : Skew normal ; Skew t ; Skew Elliptical Distributions ; Asymmetry; Value at Risk

JEL classification : C5 ;G1

I. Introduction

The Value-at-Risk (VaR), is defined as the maximum loss expected on a portfolio of assets over a certain holding period at a given confidence level (probability). It is an useful tool to estimate the exposure to market risks. Prior to present the VaR literature, it is important to consider the VaR history. We start it (see Khindanova and Rachev(1999)) in 1988 and stop it in January 1996 with the 1988 Basle Capital Accord amendment.

The year 1988 is our starting point. Originally, the Basle Capital Adequacy Accord of the G10, emphasized on credit risks (See Jackson, Maude and Perraudin (1996,1997)). Five years later, in 1993 came the European Union Capital Adequacy Directive, « EEC 6-93 » focusing on market risks. It became effective in January 1996 (See Liu(1996)). As a result, a study titled « Derivatives : Practices and Principles » in favour of VaR approaches for market risks measurement was published in July 1993 (See Kupiec(1995,1996,1997);Simons(1996);Fallon(1996);Liu(1996)). Then, in the 1994 Fisher report the Bank for International Settlements, disclosed VaR numbers (See Hendricks(1996,1997)). Otherwise, the JP Morgan RiskMetrics™ appeared in October (See JP Morgan(1995)). Later, in the Basle Committee on Banking Supervision proposal of April 1995, titled : « Supervisory Treatment of Market Risks », internal models to assess capital requirements were suggested (See Kupiec(1995,1996,1997);Jorion(1996a);Beder(1995,1997)). In the end, the 1988 Basle Capital Accord was rectified allowing standardized and internal models demanding a multiplicative factor lying between three and four for the assessment of capital requirements, in January 1996 (See Basle Committee on Banking Supervision(1996)). Keeping this short history in mind, we now have to review the VaR literature.

Bauer(2000) points out that, the modelling of financial time series generally proceeds in one of two main directions. The first is to model the process itself (this is common in the...
univariate case), and the second is to model the distribution of the price change (common in the more complicated multivariate case).

He adds that it is well known that the real distribution of the percentage price change is not Gaussian for it has fatter tails, a thinner waist than the Gaussian distribution (Zangari(1996)).

In the end, he precisesthat in the univariate case, some approaches such as ARCH-type processes in Bollerslev et al. (1992) or Levy processes based on hyperbolic distributions in Eberlein and Keller (1995) take these stylised facts into account. However, he says that, many multivariate applications, like the widely used value-at-risk concept (VaR) measuring the risk of a certain portfolio, are still based on normally distributed random variables. These distributions are considered the only ones simple enough to allow rapid computations see Longerstaey et al. (1996).

The problem is that he does not speak of asymmetry when he emphazises on nonnormality. As highlighted by Fong and Vasicek(1997), asymmetry has to be taken into account. Hence, among the two directions quoted above, this paper will take the VaR one. And it will focus on asymmetry.

About VaR, according to Gouriéroux et al.(2000) it is interesting to have a whole view of the different estimation approaches of the risk measure which are related to quantile estimation and tail analysis. Fully parametric approaches are widely used by practionners (see, e.g. JP Morgan RiskMetrics documentation), and most often based on the assumption of joint normality of asset (or factor) returns. Fully non-parametric approaches have also been proposed and consist in determining the empirical quantile( the historical VaR ) or a smoothed version of it (Harrel and Davis, 1982; Falk, 1984, 1985;Jorion, 1996;Ridder, 1997). And recently, semi parametric approaches have been developed. They are based on either extreme value approximation for the tails (Bassi et al., 1997; Embrechts et al., 1998), or local likelihood methods (Gouriéroux and Jasiak, 1999a).

Originally, the value at risk concept relies on the restrictive normality hypothesis. The problem is that this assumption is at odds with stylised facts. Consequently, as we will see below, some studies proposed leptokurtic density models, and otherwise some authors attempt to go further by taking asymmetry into account. But the models suggested are often too far from those usually taken in hypothesis in finance theory. Hence, in this study, we will rely on the distributional parametric approach testing the skew elliptical distribution model, obtained by skewing Student's $t$ and normal distributions.

The distributional approach, denotes the approach focusing on the entire distribution. To model VaR through this approach, the literature proposes unconditional leptokurtic distributions, showing high peaks and fat tails. (Zangari(1996); Longerstae(1996) ;Billio and Pelizzon(2000); Prause(1997);Bauer(2000); Khindanova and Rachev(1999) respectively on mixture of normals, Student's $t$ and switching regime; hyperbolic; and stable paretian distributions ). Other authors proposed some distributions trying to go further and take asymmetry into account (See Fong and Vasicek(1997) on gamma distribution). And finally, recent papers introduced the tail related VaR (See Danielsson and De Vries(1997a,1997b) on extreme value distributions).

So, to circumvent the normality problem, we introduce a more general class of distributions allowing for asymmetry and nearness to the usual distributions used in finance, namely the skew elliptical distributions. They can be used to assess the VaR with little
additional effort, but with much better fits to the data. The remainder of the paper is organized as follows. In section II, we present the value at risk concept. In section III, we fit the class of univariate skew elliptical distributions. It extends the standard notion of mean-variance framework. Section IV is concerned with the implementation on real data, namely returns on the CAC40 Index. And section V gathers some concluding remarks.

II. The Value-at-Risk Concept

This section basically presents the Value-at-Risk concept. We first present its definition, next come the probabilistic formulation and finally the different computational methodologies.

The Value-at-Risk (VaR), is defined as the maximum loss expected on a portfolio of assets over a certain holding period at a given confidence level (probability).

In this section, we rely on the VaR concept presented in Bauer(2000). This one is formulated as follows. Let $\Delta PV(r)$ denote the present value of a given portfolio at price $r$ of the underlying assets.

$$\Delta PV_r(x) = PV(r \cdot (1 + x)) - PV(r)$$

is the change in the value of the portfolio, if the asset price moves 100$x$%. Where $x$ is the vector of the percentual price changes of each asset and 1 is a vector containing 1 in every component. As product we use the usual vector product. The value-at-risk of level $p$ (usually $p = 5\%$ or $p = 1\%$) is defined as the infinimum value, such that

$$P(\Delta PV_r(x) \leq -VaR) \leq p$$

This implies that the probability of losing more than VaR is $p$. So the negative VaR is simply the $p$-quantile of the distribution of $\Delta PV$ does, despite its name, not represent the total risk structure of the portfolio. The distribution of delta $\Delta PV$ is usually far from the normal. Portfolios with the same VaR of level $p$ may have various conditional expected losses, if the value falls below the VaR. Presenting a VaR of a second level, for example, 5% and 1%, or categorizing $\Delta PV$ to flat, semi-heavy tailed would allow a better understanding of the risk structure.

The critical point, however, is the stochastic nature of the move $x$. Present risk management systems often suppose $x$ to be normally distributed and estimate the parameters from the past data.

Below come the different ways of computing VaR. (See Khindanova and Rachev(1999)). The VaR values are obtained from the probability distribution of portfolio value returns:

$$p = F_{\Delta PV}(-VaR) = \int_{-VaR}^{VaR} f_{\Delta PV}(x)dx,$$

where $F_{\Delta PV}(x) = \Pr(\Delta PV \leq x)$ is the cumulative distribution function (cdf) of portfolio returns in one period and $f_{\Delta PV}(x)$ is the probability density function (pdf) of $\Delta PV$. The VaR methodologies differ in ways of constructing $f_{\Delta PV}(x)$.

The traditional techniques of approximating the distribution of $\Delta PV$ are: the parametric method, historical simulation, Monte Carlo simulation and the stress-testing.
In the parametric method, the changes in portfolio value are characterized by a parametric distribution: normal or gamma. The historical simulation approach constructs the distribution of the portfolio value changes $\Delta PV$ from historical data without imposing distribution assumptions and estimating parameters. The Monte Carlo method specifies statistical models for basic risk factors and underlying assets. It simulates the behavior of risk factors and asset prices by generating random price paths. And the stress testing method examines the effects of large movements in key financial variables on the portfolio value. The price movements are simulated in line with the certain scenarios.

Empirically, this paper will preferably consider the sequence of returns $(p_{t+1} - p_t)/p_t$, instead of the price modifications $p_{t+1} - p_t$, to work with stationary observations as in Gouriéroux et al. (2000).

III. From Elliptical To Skew Elliptical Distributions


In this section we present the Sahu, Dey and Branco (2000) skew distributions class. The main contribution of their paper is the development of a new multivariate skew elliptical distributions.

The class of the elliptical distributions, introduced by Kelker (1970), includes a vast set of known symmetric distributions, for example, normal, $t$, and Pearson type II distributions. Sahu et al. (2000) propose skewed versions of these distributions which are suitable for practical implementations.

This current paper focuses on the special case of univariate processes. In univariate cases similar ideas have been studied by many authors, see for example, Aigner et al. (1977) and Chen et al. (1999).

III.1. Elliptical Distributions

Let $\sigma^2 \in R^+$ and $\mu \in R$. Consider an unidimensional random vector $X$ having probability density function (pdf) of the form

$$f(x|\mu, \sigma^2; g^{(i)}) = \left(\sigma^2\right)^{1/2} g^{(i)} \left[\left(\frac{x-\mu}{\sigma}\right)^2\right], x \in R$$

where $g^{(i)}(u)$ is a non-increasing function from $R^+$ to $R^+$ defined by

$$g^{(i)}(u) = \frac{\Gamma(1/2)}{\pi^{1/2}} \frac{g(u;1)}{\int_0^\infty r^{-1/2} g(r;1)dr}$$

where $g(u;1)$ is a non-increasing function from $R^+$ to $R^+$ such that the integral $\int_0^\infty r^{-1/2} g(r;1)dr$ exists. In this paper we shall always assume the existence of the pdf. The function $g^{(i)}$ is often called the density generator of the random vector $X$. Note that the function $g(u;1)$ provides the kernel of $X$ and other terms in $g^{(i)}$ constitute the normalizing
constant for the density $f$. In addition the function $g$, hence $g^{(i)}$, may depend on other parameters which would be clear from the context. For example, in case of $t$ distributions the additional parameter will be the degrees of freedom. The density $f$ defined above represents a broad class of distributions called the *elliptically symmetric distribution* and we will use the notation

$$X \sim El(\mu, \sigma^2; g^{(i)}),$$

henceforth in this article. Let $F(x|\mu, \sigma^2; g^{(i)})$ denote the cumulative density function (cdf) of $X$ where $X \sim El(\theta, \sigma^2; g^{(i)})$.

We consider two examples, namely the univariate normal and $t$ distributions, which will be used throughout this paper.

### III.1.1. Univariate Normal distribution

Let $g(u;k) = \exp(-u/2)$. Then straightforward calculation yields

$$g^{(i)}(u) = \frac{e^{-u/2}}{(2\pi)^{1/2}}$$

Then

$$f(x|\mu, \sigma^2; g^{(i)}) = \frac{1}{(2\pi)^{1/2} \sigma} \exp \left[-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} \right], \quad X \in \mathbb{R},$$

which is the pdf of the univariate normal distribution which mean $\mu$ and the variance $\sigma^2$. We denote this distribution by $N_i(\mu, \sigma^2)$ and the pdf by $N_i(x|\mu, \sigma^2)$ henceforth.

### III.1.2. Univariate $t$ distribution

Let

$$g(u;1,\nu) = \left[1 + \frac{u}{\nu}\right]^{-(\nu+1)/2}, \quad \nu > 0$$

Here $g$ depends on the additional parameter $\nu$, the degrees of freedom. Then straightforward calculation yields
\[ g^{(i)}(u; \nu) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu \pi)^{\nu/2}} g(u; 1, \nu) \]

Hence

\[ f(x|\mu, \sigma^2, g^{(i)}) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu \pi)^{\nu/2}} \left(\sigma^2\right)^{-\nu/2} \left[1 + \left(\frac{x - \mu}{\sigma^2}\right)^{-\nu/2}\right]^{-\nu/2}, \quad x \in R, \]

which is the density of the univariate \( t \) distribution with parameters \( \mu, \sigma^2 \) and degrees of freedom \( \nu \). We denote the distribution by \( t_{\nu}(\mu, \sigma^2) \) and the density by \( t_{\nu}(x|\mu, \sigma^2) \) henceforth.

![Figure 1](image-url)

**Figure 1:** This figure displays some elliptical curves. \( y, y_1 \) and \( y_2 \) denote three skew normal variables, with respectively delta equals to 0, -0.6 and 0.6. Sigma and mu set to 1 and 0 respectively. We can see the elliptical curves resulting when plotting them by couples.

### III.2. Skew Elliptical Distribution

The general form of the skew elliptical density function is the following,

\[ f(x|\mu, \sigma^2, \delta; g^{(i)}) = \frac{2}{\sqrt{\sigma^2 + \delta^2}} g^{(i)}\left(\frac{(x - \mu)^2}{\sigma^2 + \delta^2}\right) F\left(\frac{\delta}{\sigma \sqrt{\sigma^2 + \delta^2}} \left| 0, 1, g^{(i)} \right.\right) \]
where

\[ a = \frac{(x - \mu)^2}{\sigma^2 + \delta^2} \]

and

\[ \mu \] is the location parameter

\[ \sqrt{\sigma^2 + \delta^2} \] is the scale parameter

\[ \frac{\delta}{\sigma} \] is the shape parameter

and \( g_1^{(1)}(u) \) defined as,

\[ g_1^{(1)}(u) = \frac{\Gamma(1/2)}{\pi^{1/2}} \int_0^\infty g(a + r; 2) dr, a > 0 \]

III.2.1. Skew Normal Distribution

The general form of the skewed Normal density function, is the following

\[ f(x/\mu, \sigma^2, \delta) = 2\left(\sigma^2 + \delta^2\right)^{1/2} \phi\left(\frac{x - \mu}{\sqrt{\sigma^2 + \delta^2}}\right) \Phi\left(\frac{\delta}{\sigma} \frac{x - \mu}{\sqrt{\sigma^2 + \delta^2}}\right) \]

And the first three moments are,

\[ M_1 = \mu + \left(\frac{2}{\pi}\right)^{1/2} \delta \]

\[ M_2 = \sigma^2 + \left(1 - \frac{2}{\pi}\right) \delta^2 \]

And from the following moment generating function, we deduce the third one,

\[ M_3(t) = 2\exp\left(\mu t + \left(\frac{\sigma^2 + \delta^2}{2}\right) t^2\right) \Phi(\delta t), \]

\[ M_3 = \left( \sigma^2 + \delta^2 \right) \left[ \frac{2}{\sqrt{\pi}} (2\delta + 1) + 3\mu \right] + \mu^2 \left[ \frac{2}{\sqrt{\pi}} (\delta + 1) \right] + \mu^3 \]

**Figure 2:** From negatively to positively skewed Normal densities. With delta varying from -0.3 to 0.3, mu and sigma being set to 0 and 1 respectively

### III.2.2. Skew t Distribution

The general form of the skewed t density function, being

\[ f(x|\mu, \sigma^2, \delta, \nu) = 2(\sigma^2 + \delta^2)^{-1/2} \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\Gamma(\nu/2)(\nu\pi)^{1/2}} \left[ 1 + \frac{x^2}{\nu(\sigma^2 + \delta^2)} \right]^{-\nu/2} \]

\[ T_{\nu+1} \left[ \left( \frac{\nu + q(x_*)}{\nu + 1} \right)^{\frac{1}{2}} \frac{\delta}{\sigma} \frac{x_*}{\sqrt{\sigma^2 + \delta^2}} \right] \]

with

\[ q(x_*) = \left( \sigma^2 + \delta^2 \right)^{-1} x_*^2 \]

and

\[ x_* = x - \mu \]

And the first three moments are,
\[ M_1 = \mu + \left( \frac{\nu}{\pi} \right)^{1/2} \frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} \delta, \]

\[ M_2 = \left( \sigma^2 + \delta^2 \right) \frac{\nu}{\nu - 2} - \frac{\nu}{\pi} \left( \frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} \right)^2 \delta^2, \text{ when } \nu > 2 \]

And the third one can be approximated by,

\[ M_3 = M_1 \left\{ \frac{\nu(2\nu - 3 + \delta^2)}{(\nu - 2)(\nu - 3)} - 2 \right\} \]

**Figure 3:** From negatively to positively skewed Student's \( t \) densities. With delta varying from -0.6 to 0.6, mu and sigma being set to 0 and 1 respectively.
IV. Data and Estimation

The data series used is the CAC40 Index. Three years of daily observations from 30 April 1998 to 30 May 2001 are used in the analysis, providing a total of 804 observations.

![Skew Normal and Skew t PDF Tails](image)

**Figure 4:** The CAC40 Index returns fitted by the Skew $t$ probability density function, displays a fatter left tail than both the Gaussian and the Skew Normal probability density functions.

Below in Table 1 are gathered the empirical results at 1% and 5% of VaR from the skew elliptical model.

**Table 1: VaR Estimates on the CAC40 Index using the Skew Elliptical Model**

<table>
<thead>
<tr>
<th></th>
<th>Skew $t$</th>
<th>Empirical</th>
<th>Skew $N$</th>
<th>Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 1%$</td>
<td>0.0444</td>
<td>0.046666</td>
<td>0.0356</td>
<td>0.0364</td>
</tr>
<tr>
<td>$p = 5%$</td>
<td>0.0248</td>
<td>0.025033</td>
<td>0.0244</td>
<td>0.025</td>
</tr>
</tbody>
</table>

Obviously the 5% Value at Risk is quite well estimated by both the Skew $t$ and the Gaussian models. The Skew Normal underestimates it. And, the 1% VaR is only correctly estimated by the skew $t$ model. Hence, at both levels, this latter outperforms the other probability models.

For the CAC40 Index returns $\{X_i\}_{i=1,...,n}$ we estimate the density function $f$ using a Picard type Kernel. This kernel density is constructed as follows,

$$f(X) = \frac{1}{n} \sum_{i=1}^{n} K_\rho(X_i - X)$$

$$K_\rho(X) = \frac{1}{\rho} e^{\rho |x|} \text{ with } \rho = 0.625.$$

The figure below, displays the CAC40 Index Picard type kernel density. We can notice leptokurtosis, fat tails and negative skewness in the returns.
Figure 5: The Picard Type kernel density. Where \( n \) and \( \rho \) are respectively the sample size and the bandwidth size. The kernel density of the CAC40 Index returns, displays leptokurtosis fat tails and negative skewness.

In addition to the asymmetry, the skew elliptical model has another advantage. It contains the elliptical density model as a limiting case. If \( \delta \) is close to zero, the skew \( t \) and the skew normal become respectively Student's \( t \) and normal.

Figure 6: This figure displays the skew elliptical probability density functions fitted to the CAC40 Index returns. The Skew \( t \) pdf is plotted with \( \mu = 5.809E-03, \sigma = 1.187E-02, \delta = -5.597E-3, \nu = 5. \) The Skew Normal is plotted with \( \mu = 6.259E-04, \sigma = 1.548E-02, \delta = -1.253E-11. \) And the benchmark Gaussian pdf is plotted with \( \mu = 0.57E-03, \sigma = 1.580E-02. \) The Skew \( t \) probability density function seems more leptokurtic and fatter tailed than both the Gaussian and the Skew Normal ones.
We can see that at the different levels 5% and 1%, the VaR is underestimated by both the Gaussian and the Skew Normal models. To be able to judge the risk of a portfolio one needs more than just the VaR on only one level, for example, a second VaR, a category like heavy-tailed or flat-tailed or even the whole risk structure. Using the VaR estimated on Gaussian random variables will result in a misrepresentation of the real risk. Whereas the skew $t$ VaR fits correctly the empirical one.

**Figure 7:** The Skew $t$ VaR better is closer to the empirical VaR than the Skew Normal and the Gaussian ones. The Skew Normal and the Gaussian ones underestimate the empirical Value-at-Risk.

**Figure 8:** The Skew Normal and the Gaussian VaR clearly under the Skew $t$ VaR at any level and close to the Gaussian one.
Figure 9: The Skew t VaR seems the closest to the empirical VaR. This probability density function appears to be the best in accuracy

V. Concluding Remarks

In this study, the first step consisted in presenting the Value-at-Risk concept which relies on the restrictive normality hypothesis. Because of its violation, many authors proposed elliptical leptokurtic density models. But these latter don’t take asymmetry into account. Moreover, the distribution candidates suggested are always too far from those usually taken in hypothesis in finance theory.

Consequently, we suggested the Skew Elliptical density functions as an alternative model to the elliptical family functions.

Finally, our results are consistent with empirical facts. The skew t distribution occurs to fit financial data better than the both the Gaussian and the Skew Normal ones. Calculating the VaR on the basis of Skew t density gives more precise results for all levels $p$ simultaneously. Considering both the Gaussian and the Skew Normal distributions, the resulting estimators underestimate the empirical one at every level. Compared to the Skew t probability density function, these probability density functions do not take the returns density leptokurtosis and fat tail into account.

The direct goal of this paper was an attempt to extend the mean-variance value at risk computation to higher order moments, and particularly asymmetry from the skew elliptical density family functions.

Further work on this topic should propose a multivariate form of the value at risk measure.

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