# Pricing Prepayment Option
# In C&I Loans at Origination

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**Comments Welcome**

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## Abstract

This paper presents a pricing model of commercial and industrial (C&I) loan prepayment option. Business borrowers make rational decisions to refinance when interest rates fall or their credit standing improves. Business borrowers also prepay with internal equity to optimize capital structure and reduce financing costs. Prepayment value in C&I loans reflects the option value depending on the market conditions, borrower's creditworthiness and the need to change capital structure and reduce costs. In this paper, we propose a contingent-claims approach to pricing C&I loan prepayment contract, given loans being defaultable, and transaction costs and fees being proportional to loan amount. The prepayment option value is determined via a double non-recombining tree. The model differs from the mortgage prepayment models by fitting observed business borrower prepayment behavior differently.

# Pricing Prepayment Option in C&I Loans at Origination

## 1. Introduction

Modeling of prepayment is essential in pricing mortgage contracts[1], mortgage-backed securities[2], and C&I loans, since prepayment truncates the timing and amount of expected cash flows. Even though the prepayment of mortgagors has been widely researched over the past 20 years, the prepayment of business borrowers has not attracted the same degree of attention. In this paper, we ask how business borrowers differ from individuals[3] when prepaying their debts and how to determine the prepayment option value at origination. These questions are important because C&I loan prepayment is hardly a trivial issue. The Fed's survey of terms of business lending states that prepayment penalties are becoming more common on larger loans of above $1 million. In fact, the February 4-8, 2002 statistical release surveys that 32.8% of the total $93,387 million of C&I loans made by commercial banks are subject to prepayment penalty. Prepayment means added-value to the borrower and increasing cost to the lender when replacing the loan into the lending portfolio. Pricing (correctly) or not signifies power shifts among participants in the general credit market.

The behavior of business prepayment differs significantly from that of individual mortgagors. Mortgagor prepayment behavior has been documented as highly irrational. Many borrowers prepay their mortgages when current mortgage rates rise above their coupon rates and fail to prepay when the prepayment option is substantially in the money. Various research (i.e. Hayre, Chaudhary and Young 2000, Green and LaCour-Little 1999, Abrahams 1997 ) accounts this irrationality with exogenous factors such as home sales, relocation, seasonality, diversity in borrower types, etc. In contrast, business borrowers are more homogeneous in type, more sensitive to changes in the refinancing environment, and prepay in a rational way[4]. Business borrowers who are expected to maximize owner/shareholder value, look constantly to minimize financing costs. In that sense, business borrowers will "ruthlessly" exercise

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[1] See LeRoy (1996) for mortgage valuation under prepayment.

[2] See Kang and Zenios (1992) for a discussion of fix-rate mortgage under prepayment.

[3] By individuals, we refer to mortgagors or individual families.

[4] Abraham and Theobald (1997) study the difference between commercial and individual mortgagors, which postulates that commercial mortgagors are more rational prepayers. We extend that notion directly to business borrowers.

an option given the right refinancing circumstances. The cross-collateralization[5] of multiple assets enables business borrowers to pay off loans when the need to change capital structure arises. Business and mortgage borrowers have different incentives to prepay which should be modeled differently in order to capture the benefit of refinancing and full payoffs.

Several researchers have proposed models of mortgage prepayment based on empirical studies. By fitting past prepayment rates with a set of explanatory variables, a prepayment function can be derived to explain current and future prepayment behaviors. Schwartz and Torous (1989), Charlier (2001), Charlier and van Bussel (2001), and Jegadeesh and Ju (2000) are examples using this approach. However, the accuracy of these reduced-form representations for the purpose of forecasting the future out of the sample needs to be studied further. Another strand of research has been based on the contingent-claims approach. Dunn and McConnell (1981a,b) model prepayment without costs on prepayment. Their research has been extended notably in Stanton (1995) , Archer and Ling (1993), by explicitly modeling heterogeneous transaction costs. Other works following this strand are by Kau, et al. (1992), Deng, Quigley and van Order (1995), and Ambrose and Sanders (2001) who extend the option pricing framework to model the competing risks of default and prepayment encountered by lenders. Unlike the realistic economic modeling in mortgage market, there is no clear consensus on the modeling of business loans prepayment. We follow the contingent-claims approach and propose a simple modeling of the more rational behavior of business loans prepayment.

Prepayment is usually priced as an American call option on a defaultable debt owned by the borrower. Hilliard and Kau (1998) valued prepayment and default options in a fixed-rate mortgage using a bivariate binomial lattice, with real-estate value and spot interest rate as the two underlying state variables. The same options were valued by Kau et. al. (1992) via a two-state explicit finite-difference technique. The model presented in this paper differs from Hilliard and Kau (1998) in many ways. First, business borrowers prepay with different incentives; second, the two underlying state variables are forward credit spread and forward interest rate, developed by Heath, Jarrow and Morton (1990, 1992)

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[5] For its role in analyzing CMBS, see Sanders (1999).

(HJM); third, the proposed model is a double non-recombining tree to capture the dynamics of credit spreads and forward interest rates simultaneously. The double non-recombining tree arbitrage-free framework is an extension of the double bivariate lattice approach described by Das and Sundaram (2000). Their framework has been extended in many ways to price prepayment options in C&I loans. Notably, time-varying volatility structures[6], multiple interest payments, lockout provisions, and transaction costs are modeled to derive the true value of prepayment option embedded in C&I loans.

The objective of this paper is twofold: first rationalize the prepayment behavior of business borrowers; second price C&I loan prepayment option given loans being interest-bearing, defaultable, and transaction costs and fees being proportional to the loan amount. The prepayment option value is determined via a double non-recombining tree, which assumes that business borrowers make prepayment decisions at discrete intervals before the maturity of the loan. The lockout provision is also modeled without difficulty. The remainder of the paper is organized as follows: Section II analyzes the nature of prepayment as an option and describes how business prepayment differs from mortgage prepayment in terms of prepaying behavior. Section III lays out the model by first providing assumptions underlying the model. Second modeling the market value of the defaultable interest-bearing loan; third modeling book value of the defaultable interest-bearing loan; and last modeling prepayment options on various defaultable loans. Section IV describes more specifications on the implementation of the model. Section V illustrates a detailed implementation by pricing prepayment option in a floating-rate loan contract and shows how different parameters may influence the prepayment option value. Section VI provides concluding remarks.

## 2. A Closer Look At Prepayment

The ongoing debate[7] on the legitimacy of prepayment penalties in the mortgage market has resulted in the introduction of anti-predatory-lending bills in many states and district legislatures in the

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[6] Both exponentially dampened volatility and exponentially dampened volatility proportional to the spot rate avoid negative interest rates.

[7] See Scriber (2001), Andrews (2001), Paletta (2002).

United States. By law, some states allow early repayment without penalties. In other states, however, the law allows a lender to charge a penalty when a borrower reduces the balance or pays back a loan before maturity. Consumer advocates criticize that the prepayment penalty is an abusive or predatory lending practice that should be prohibited, while the lending community contends that the prepayment provision is a boon to borrowers. So what's the true nature of prepayment?

## 2.1 True Nature of Prepayment: Boon or Burden?

In prior years, the lending industry has followed the "rules of thumb" to price prepayment penalties. Rule 1, a prepayment penalty amounts to a portion of the total loan or the loan balance, for example, 3% of the loan amount or 5% of the loan balance; Rule 2, a prepayment penalty amounts to a portion of the future interest, for example, six months interest; Rule 3, a prepayment penalty follows a declining structure, for example, the penalty of a ten-year loan is equal to 100% of one year's interest if a borrower prepays in the first year of the contract, declining by 20% per year to zero after five years. Is this penalty anticompetitive? Why should a borrower be penalized for paying off a loan? Is it fair to have to pay interest on principal that one has never touched? These philosophical questions suggest that the rationale for these pricing rules is less than convincing.

For a borrower, prepayment provides reinvestment flexibility when market situations change. Actually, the borrower has bought a call option on the value of the loan from the lender at origination. A prepayable loan can be interpreted as equivalent to a nonprepayable loan plus a call option. Unlike a financial option, this real option has not been paid until being exercised later. Prepayment penalties should be viewed as fees in arrears reflecting the value of a call option at origination, rather than interest on unused principal. If a borrower pays off the loan early, he/she will have to pay the price of the call option owed, which is mistermed as a "penalty" and mispriced with inaccurate "rules of thumb." Owning a prepayment option or not is the choice of the borrower at the time of signing the contract, a boon to the borrower; but the lender should price the option correctly to be fair, otherwise the boon becomes a burden. Working with "rules of thumb" is simply unfair and counter-intuitive.

Despite many calls for further research on the embedded optionality in C&I loans[8], the loan prepayment option has rarely been estimated and priced in a systematic way. As we can see from the Fed's survey, more than 70% of the option embedded in the loans tend to be given away. This lies in the fact that many loans are repriced through time in response to variations in the interest rate, ratings or creditworthiness. Therefore, the value of refinancing prepayment options associated with a completely repriceable lending agreement will vanish as rates, both interest rates and credit spreads, are reevaluated from period to period. However, the non-refinancing prepayment which cannot be diverted away with repricing still exists. Increasing competition in the C&I loan market seems to have forced lenders to give away the option with calculated risk. Nevertheless, the option could have also been given away without realizing its true value to the borrower or the true cost to the lender replacing the loan into the lender's portfolio. The prepayment option has been wrongly understood in the mortgage market and undervalued in the C&I loan market. In order to evaluate the prepayment option, we have to first characterize the borrower's prepayment behavior.

## 2.2 Loan Prepayment: Borrowers' Behavior and Incentives

A loan contract may be terminated prematurely by either default or prepayment[9]. Business borrowers and individual mortgage borrowers face different default options as they possess different financial might. Mortgage borrowers tie their investment in real estate assets, and own few opportunities to substitute one asset class for another[10]. On the other hand, business borrowers normally own multiple assets which enable cross-collateralization. That is, the cash flows on several assets can be used to make loan payment on one nonperforming asset and this pooling effect reduces the risk of default for business borrowers. Default is an extreme event, and prepayment out of default or recovery is modeled indirectly in the subsequent sections.

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[8] See, for example, Aguais and Santomero (1997), and Aguais, Forest and Rosen (2000).

[9] Prepayment can also be initiated by the lender if certain covenants are breached.

[10] This is true for single family borrowers, but less true for commercial real estate investors as pointed out by Abraham and Theobald (1997).

A rational borrower's objective is to maximize the present value of his or her investment. With refinancing opportunities spreading over the expected tenure, it is rational for the borrower to refinance the old contract with a new one once the interest rate falls[11] or credit standing improves. Regardless of other factors influencing the prepayment, both individuals and business borrowers should behave rationally to optimize their wealth. However, not all mortgage borrowers reach the same refinancing decision when refinancing conditions are in the borrowers' favor. This irrationality in the mortgage market, in-the-money non-calls[12], has been well studied to account for heterogeneous prepayment speed and transaction cost such as in Giliberto and Ling (1992). Business borrowers, who are expected to maximize owner/shareholder value, ought to look constantly to minimize financing costs. In that sense, business borrowers will "ruthlessly" exercise an option given the right refinancing circumstances. Abraham and Theobald (1997) report that commercial mortgages borrowers behave more "ruthlessly" than single family borrowers when exercising prepayment options and we extend this notion directly to general business borrowers. There are times when business borrowers are reluctant to exercise an in-the-money option because that would damage relationship with banks. A little gain today might make future borrowing difficult. We view this relationship concern as part of transaction costs, that will increase the exercise price just like the necessary fees to search for a new loan do. Thus, by introducing transaction costs into the model, we assume away this in-the-money non-calls irrationality for business borrowers.

Prepayment can also be induced by non-financial factors. A phenomenon observed often in the mortgage market is that people refinance when current mortgage rates rise above their coupon rates, namely, out-of-the-money calls. Mortgage borrowers prepay for various non-financial reasons[13], such as job relocation, divorce, or death in the family that lead to house sale. These non-financial factors driving mortgage prepayment result in irrational refinancing with higher rates. Non-financial factors change over time in unpredictable ways, which adds significant difficulties to characterize prepayment behavior

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[11] See, for example, Green and Shoven (1986).

[12] By non-calls, we mean that option owner doesn't exercise the option even if it's in the money.

[13] See Dunn and McConnell (1981a,b) and Kau et al. (1992). It adds non-interest-rate driven prepayments to the optimal call model.

of mortgage holders. However, business borrowers don't share the same non-financial prepayment incentives. Refinancing in the spot market voluntarily with higher rates is not expected from a rational business borrower. Even though cross-collateralization reduces the possibility of being forced into refinancing with higher rates, this may happen to a financially weak borrower when he/she broke a debt covenant and the lender demanded immediate repayment of the debt. Without sufficient cash flows to cover the unexpected payment, the borrower may have to refinance with higher rates. Empirical studies show that lenders usually do not demand an immediate payment when the borrower's financial situations change adversely or certain covenants are breached. Dichev and Skinner (2001) find that technical violations occur in about 30 percent of a total of 8804 loans, but violations are not necessarily associated with financial distress and violations are often waived for healthy firms. For unhealthy firms, the violation of a loan covenant is merely an occasion for renegotiation and as an extreme, demanding for immediate payment rarely happens. Therefore, we can view this irrational prepayment as an anomaly and exclude it from our analysis.

Another situation is intentional neglect: some mortgage borrowers prefer to pay more than scheduled or pay off mortgages with inside equity, regardless of the going rates. In this way, borrowers can force savings and build up home equity faster, as recognized by Hayre, Chaudhary and Young (2000). Similar to prepayment financed with home equity as what we find with individual mortgagors, business borrowers do prepay with internal equity to reduce debt, or as an alternative, the project under the loan has been liquidated and the loan has been repaid out of the firm's own excess cash flows. Business borrowers do pay off debt without refinancing the old debt in the spot market for a reason. The non-optimal behavior in view of the market conditions is rightfully optimal in light of the savings made by paying off the loan. This non-financial prepayment incentives follow the tractable optimization rule, which is modeled in Section III.

Business borrowers exercise an in-the-money option ruthlessly. If we introduce transaction costs into the modeling, in-the-money non-calls can be easily controlled. This makes refinancing prepayment a rational decision by comparing the book value of the loan with the market value of the loan, netting of transaction costs. Out-of-the-money calls are rare events that may be excluded from analysis. Prepaying with internal cash flows is optimal when the borrower can save interests on idle loans. Thus, irrationality is no longer the factor bothering modeling which makes the optimal call model an attractive solution to C&I loan prepayment.

# 3. The Pricing Model

Business prepayment option value depends on several factors: interest rates, credit spreads, transaction costs and fees, and internal demands to change capital structure. We can categorize C&I loan prepayment into two classes: refinancing and non-refinancing, and both of them are optimal calls. Refinancing prepayments are optimal calls on interest rates and credit spreads with transaction costs. It is the market value of the existing loan to be refinanced with lower interest rate and/or narrower credit spread less its book value, netting for transaction costs and fees. Non-refinancing prepayments are optimal calls on risk-free rate when business borrowers pay off the loan with internal resources; It is the market value of the existing loan to be paid off with risk-free rate less its book value. Non-refinancing option is more valuable as the borrower's own cash flows are free of credit risk and transaction costs. The prepayment option value is the sum of the two with a proper balance of occurrence probabilities.

## 3.1 Assumptions

The model is developed in discrete time rather than in continuous time. This preference is due to the American style of the option which requires the optimal stopping rules, time-varying transaction costs, and more accurate description of multiple interest payments. The continuous time solutions are not accurate to this structure and are ill-fitted to the problem. The principal assumptions are these:

– A finite time interval $[0, T^*]$ is divided into subperiods of equal length $\Delta$. The discrete periods correspond to the times $0, \Delta, 2\Delta, ..., T^*$. Let $j\Delta$ denote a typical time-point $t$.

– The loans studied can be of fixed-rate or floating-rate, repriceable, completely non-repriceable, or partially non-repriceable, defaultable, prepayable and interest-bearing.

– The sources of uncertainty in the economy are the forward interest rate and forward credit spread, following discrete jump processes free of arbitrage opportunities. The forward-rate and spread processes incorporate risk-neutral drifts which are derived in terms of volatilities as in the HJM. Volatilities may follow different structures: constant, time-varying, exponentially dampened, and exponentially dampened proportional to the spot rates.

– There is an exogenously specified probability $\pi$ that non-refinancing prepayment occurs, and consequently, refinancing prepayment occurs with a probability of $1 - \pi$[14].

– Business borrowers are utility maximizers who prepay optimally to gain from lower interest rates and narrower credit spreads. They also act optimally to reduce debt to avoid financial distress or simply to get rid of excess cash. Partial prepayment or curtailment is assumed too trivial to be considered for business borrowers.

– Refinancing transaction costs include a fixed portion representing origination costs $TC_f(t)$ and a variable portion $TC_v(t)$ depending on the loan amount $L$. Transaction costs $[TC(t) = TC_f(t) + TC_v(t)]$ increase prepayment option exercise price so as to decrease the option value. Non-refinancing prepayment does not incur transaction costs due to the internal dealing nature.

– Markets are free of arbitrage, so there exists a risk-neutral measure under $Q$; all references to randomness and expectations are with respect to $Q-$martingale.

Based on the above assumptions and analysis, the general formula for the prepayment option is

$$Prepayment\ option\ value = \pi \cdot non-refinancing\ option\ value + (1-\pi) \cdot refinancing\ option\ value. \tag{0.1}$$

Let $\tau^*$ be an expiration date on the time interval $[0, T^*]$ , the above formula may be simplified as

$$A(\tau^*) = \pi \cdot \Lambda(\tau^*) + (1 - \pi) \cdot \Gamma(\tau^*). \tag{0.2}$$

## 3.2 Market Value of Interest-Bearing Loans

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[14] We leave out prepayment resulting from default and bankruptcy as its occurance probability is rather small.

Duffie and Singleton (1999) show that a defaultable claim may be priced like a default-free claim by replacing the risk-free interest rate process with a risk-adjusted process, which is a function of the risk-free rate plus default hazard rate $h$ multiplied by $l$, the expected fractional loss in market value. This risk-adjusted process accounts for the probability and timing of a default and the effect of losses on default. The product of the default hazard rate and fractional loss $h \cdot l$, does not depend on the value of the defaultable claim itself, and it may be exogenously specified. By parameterizing the risky rate instead of the risk-free rate, we have a convenient framework to the pricing of corporate debt. Even though the hazard rate and the fractional loss can be identified separately, normally the product of the two $hl$ enters the pricing formula together through the mean-loss rate process. We thereafter transform $hl$ into a credit spread process[15].

Let $f(t, T)$ denote the forward rate on the default-free debt applicable to the period $(t, T)$. Note that time follows this relationship: $0 \leqslant t \leqslant \tau^* \leqslant T \leqslant T^*$. When $t = T$, the rate $f(t, T)$ is the "short rate" and denoted by $r(t)$. The forward rate process is

$$f(t + \Delta, T) - f(t, T) = \alpha(t, T)\Delta + \sigma(t, T)X_1\sqrt{\Delta}, \tag{0.3}$$

where $\alpha$ is the drift and $\sigma$ is the volatility of the process; and $X_1$ is a random variable.

Let $\varphi(t, T)$ denote the defaultable forward rate on the defaultable debt implied from the spot yield curve. The forward credit spread $s(t, T)$ on the defaultable debt is defined as $s(t, T) = \varphi(t, T) - f(t, T)$, which evolves according to the process

$$s(t + \Delta, T) - s(t, T) = \beta(t, T)\Delta + \eta(t, T)X_2\sqrt{\Delta}, \tag{0.4}$$

where $\beta$ and $\eta$ are the drift and volatility coefficients respectively, and $X_2$ is a random variable. Credit spreads on the defaultable debt represent the costs of default, which depend on the probability of default and recovery rate. By construction, this credit spread process is consistent with the default hazard rate $h$ and fractional loss $l$[16].

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[15] This approach is widely recognized as "reduced-form" approach to credit risk. For more models on credit risk, we refer to Cossin and Pirotte (2001), and Bielecki and Rutkowski (2002).

[16] See Appendix 1 for a proof following Duffie and Singleton (1999).

Let $P(t,T)$ denote the time $t$ price of a default-free zero-interest loan of maturity $T \geqslant t$, and $P^*(t,T)$ a defaultable zero-interest loan. Then we have

$$P(t,T) = \exp\left\{-\sum_{j=t/\Delta}^{T/\Delta-1} f(t,j\Delta)\cdot\Delta\right\}, \tag{0.5}$$

and

$$P^*(t,T) = \exp\left\{-\sum_{j=t/\Delta}^{T/\Delta-1} \varphi(t,j\Delta)\cdot\Delta\right\} = \exp\left\{-\sum_{j=t/\Delta}^{T/\Delta-1} [f(t,j\Delta)+s(t,j\Delta)]\cdot\Delta\right\}. \tag{0.6}$$

We define an interest-bearing loan to be a sequence of discrete cash flows $C_1, C_2, ..., C_{T/\Delta}$ at times $\Delta \leqslant 2\Delta \leqslant \cdots \leqslant T$. The first $T/\Delta - 1$ payments represent expected discrete interests, and the last payment, at time $T$, represents an interest payment plus principal, that is $C_{T/\Delta} = C + L$. Thus, the market value of a default-free interest-bearing loan can be viewed as equivalent to a portfolio of default-free zero-interest loans, and the market value of a defaultable interest-bearing loan is a portfolio of its defaultable counterpart:

$$D(t) = \sum_{j=t/\Delta+1}^{T/\Delta} E^t\left[C_j\right]\cdot P(t,j\Delta), \tag{0.7}$$

and

$$D^*(t) = \sum_{j=t/\Delta+1}^{T/\Delta} E^t\left[C_j\right]\cdot P^*(t,j\Delta). \tag{0.8}$$

A completely repriceable loan is equivalent to $D^*(t)$ as the loan is to be repriced through time in response to the changes in the credit standing and interest rates. A non-repriceable loan can be either a fixed-rate or floating-rate loan with floating-rate loan being partially repriceable[17]. We need to note that the time $t$ price of the interest-bearing loan does not include the interest payment made at time $t$ because the payment is the interest owed for the past period $j-1$. As a convention, the loan price represents the price ex-interest.

## 3.3 Book Value of Interest-Bearing Loans

Let $f$, $s$ be the risk-free rate and credit spread fixed in the loan contract. A fixed-rate loan bears an interest rate of $f+s$ unchanged throughout the tenure, while a floating-rate loan bears a floating interest rate of $f(t,T)$ with credit spread remaining fixed at $s$. Let $M^*(t,T)$ denote the time $t$ price

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[17] That is, the indexed rate is floating and the credit spread is fixed.

of a defaultable fixed-rate zero-interest loan of maturity $T \geqslant t$, and $N^*(t,T)$ denote its floating-rate counterpart. We have the book value of the two loans as

$$M^*(t,T) = \exp\left\{-(f+s)\cdot(T-t)\right\}, \tag{0.9}$$

and

$$N^*(t,T) = \exp\left\{-\left[\sum_{j=t/\Delta}^{T/\Delta-1}\left[f(t,j\Delta)+s\right]\cdot\Delta\right]\right\}. \tag{0.10}$$

Therefore, the book value of a defaultable fixed-rate interest-bearing loan at time $t$, as denoted by $U^*(t)$, is defined as

$$U^*(t) = \sum_{j=t/\Delta+1}^{T/\Delta} E^t\left[C_j\right]\cdot M^*(t,j\Delta), \tag{0.11}$$

and the book value of a defaultable floating-rate interest-bearing loan at time $t$, as denoted by $V^*(t)$, is simply

$$V^*(t) = \sum_{j=t/\Delta+1}^{T/\Delta} E^t\left[C_j\right]\cdot N^*(t,j\Delta). \tag{0.12}$$

## 3.4 Prepayment Options

Recall the general formula for a prepayment option is

$$A(\tau^*) = \pi\cdot\Lambda(\tau^*) + (1-\pi)\cdot\Gamma(\tau^*). \tag{0.13}$$

For a non-refinancing option, the exercise price is the book value of the loan. The exercise price for a refinancing option is the book value of the loan plus transaction costs, both of them may change across time. A prepayment decision may be made at any discrete node, resembling the American feature that the option can be exercised at any date prior to maturity. The standard procedure for pricing an American call option is backward induction, a procedure also known as stochastic dynamic programming. We start at the expiration date of the option, work backward, and compare the value of immediate exercise and continuation for optimal exercise until time $t$. We illustrate this concept with a prepayment option on a completely repriceable loan.

### Prepayment option on a completely repriceable loan

The book value of a completely repriceable loan, depending on how frequent the loan may be repriced, is approximate to its market value. In that sense, there is no option value to refinance the loan with changed rates as the borrower will have to incur transaction costs without gains. The prepayment option value lies in non-refinancing prepayment. Either because of liquidation of the underlying project or change of capital structure, the option is valuable to the borrower. Lender will have to replace the loan into the lending portfolio which will incur costs. So prepayment option on a completely repriceable loan must be valued to reflect the benefit to the borrower and the cost to the lender. If the loan is paid off out of the borrower's own cash flows, the value of the loan in borrower's eyes at time $t$ is that of the default-free interest-bearing loan $D(t)$. Its exercise price is the book value of the defaultable loan, approximated with $D^*(t)$. At the expiration date $\tau^*$, given not exercised before $\tau^*$, the prepayment option value on a completely repriceable loan is

$$A(\tau^*) = \pi \cdot \Lambda(\tau^*) + 0. \tag{0.14}$$

Leave out $\pi$, we consider the value of the non-refinancing prepayment option $\Lambda(t)$ at the expiration date $\tau^*$:

$$\Lambda(\tau^*) = \max\left[D(\tau^*) - D^*(\tau^*), 0\right]. \tag{0.15}$$

If it is not exercised at time $\tau^* - \Delta$, its value is

$$E^{\tau^*-\Delta}\left[\Lambda(\tau^*)\right] \cdot P(\tau^* - \Delta, \tau^*). \tag{0.16}$$

The time $\tau^* - \Delta$ does not include the interest payment at time $\tau^*$, for $D(\tau^*)$ and $D^*(\tau^*)$ are the ex-interest prices of the loan. If the option is exercised at time $\tau^* - \Delta$, its price is $D(\tau^* - \Delta) - D^*(\tau^* - \Delta)$. Please note that this exercised value includes the present value of the interest payment to be received at time $\tau^*$. The American option value at time $\tau^* - \Delta$ is the larger of the two:

$$\Lambda(\tau^* - \Delta) = \max\left\{E^{\tau^*-\Delta}\left[\Lambda(\tau^*)\right] \cdot P(\tau^* - \Delta, \tau^*),\ D(\tau^* - \Delta) - D^*(\tau^* - \Delta)\right\}. \tag{0.17}$$

Moving backward to time $\tau^* - 2\Delta$, the value of the prepayment option is

$$\Lambda(\tau^* - 2\Delta) = \max\left\{E^{\tau^*-2\Delta}\left[\Lambda(\tau^* - \Delta)\right] \cdot P(\tau^* - 2\Delta, \tau^* - \Delta),\ D(\tau^* - 2\Delta) - D^*(\tau^* - 2\Delta)\right\}. \tag{0.18}$$

Continuing to time $t$ by backward induction, we deduce the value of the non-refinancing prepayment option as

$$\Lambda(t) = \max\left\{E^t\left[\Lambda(t+\Delta)\right]\cdot P(t,t+\Delta),\ D(t) - D^*(t)\right\}. \tag{0.19}$$

Adding back the probability measure $\pi$, we derive the value of a non-refinancing prepayment option as

$$A(t) = \pi\cdot\max\left\{E^t\left[\Lambda(t+\Delta)\right]\cdot P(t,t+\Delta), D(t) - D^*(t)\right\}. \tag{0.20}$$

### Prepayment option on a non-repriceable loan: fixed-rate case

A fixed-rate loan is not repriceable which will probably lead to both refinancing and non-refinancing prepayment. In case of a refinancing prepayment, the market value of a defaultable fixed-rate interest-bearing loan at time $t$ is $D^*(t)$; the book value of its counterpart is $U^*(t)$; and the borrower will incur transaction costs $TC(t)$ for searching and negotiating new loan in the market. For a non-financing prepayment, the book value of this loan remains to be $U^*(t)$. As analyzed above, non-refinancing prepayment will be financed with internal cash flows which are riskless and free of transaction costs. Thus, at the expiration date $\tau^*$, the prepayment option value on a fixed-rate loan, given not exercised before $\tau^*$, is

$$\begin{aligned} A(\tau^*) &= \pi\cdot\Lambda(\tau^*) + (1-\pi)\cdot\Gamma(\tau^*) \\ &= \pi\cdot\max\left[D(\tau^*) - U^*(\tau^*), 0\right] + (1-\pi)\cdot\max\left[D^*(\tau^*) - U^*(\tau^*) - TC(\tau^*), 0\right] \end{aligned} \tag*{0.21a}$$

The prepayment option value at time $t$ by backward induction is

$$\begin{aligned} A(t) &= \pi\cdot\max\left\{E^t\left[\Lambda(t+\Delta)\right]\cdot P(t,t+\Delta),\ D(t) - U^*(t)\right\} \\ &\quad +(1-\pi)\cdot\max\left\{E^t\left[\Gamma(t+\Delta)\right]\cdot P(t,t+\Delta),\ D^*(t) - U^*(t) - TC(t)\right\}. \end{aligned} \tag*{0.22}$$

### Prepayment option on a partially non-repriceable loan: floating-rate case

A floating-rate loan is partially non-repriceable, i.e., its credit spread has been fixed in the contract. Readily, if it is a refinancing prepayment, the market value of this loan at time $t$ is $D^*(t)$, the book value is $V^*(t)$, and transaction costs $TC(t)$ apply. A non-refinancing prepayment implies the loan to be a default-free loan of $D(t)$ and its book value is $V^*(t)$. The prepayment option value at time $t$ can be easily derived as

$$\begin{aligned} A(t) &= \pi \cdot \max\left\{E^t\left[\Lambda(t+\Delta)\right] \cdot P(t,t+\Delta),\, D(t) - V^*(t)\right\} \\ &\quad +(1-\pi) \cdot \max\left\{E^t\left[\Gamma(t+\Delta)\right] \cdot P(t,t+\Delta),\, D^*(t) - V^*(t) - TC(t)\right\}. \qquad 0.23 \end{aligned}$$

### Lockout provision[18]

A lockout provision in a loan contract prohibits any prepayment prior to some future date $t^*$. We need to restate the time relationship as $0 \leqslant t < t^* \leqslant \tau^* \leqslant T \leqslant T^*$. This feature resembles that of a delayed-exercise American call option. With a lockout provision, prepayment can only occur between time $t^*$ and $\tau^*$. Let $K^*$ denote a number to be added onto the exercise price. So the exercise schedule for the entire life of the option $t \leqslant i \leqslant \tau^*$ can be given separately as

$$K^* \equiv M,\ for\ t \leqslant i < t^* \qquad 0.24$$

$$K^* \equiv 0,\ for\ t^* \leqslant i \leqslant \tau^* \qquad 0.25$$

where $M$ is a large positive constant to make exercise non-optimal between time $t$ and the lockout date $t^*$, even if it allows exercise. While during the exercise period, $K^*$ equals $0$ which has no impact on the option value. For example, given a prepayment option on a completely repriceable loan, we have

$$A(t) = \pi \cdot \max\left\{E^t\left[\Lambda(t+\Delta)\right] \cdot P(t,t+\Delta),\, D(t) - D^*(t) - K^*\right\}. \qquad (0.26)$$

Prior to time $t^*$, the maximum on the right hand side is always $E^t\left[\Lambda(t+\Delta)\right] \cdot P(t,t+\Delta)$, because $D(t) - D^*(t) - M$ is a large negative number. After $t^*$ the maximum goes back to $\max\left\{E^t\left[\Lambda(t+\Delta)\right] \cdot P(t,t+\Delta),\, D(t) - D^*(t)\right\}$, as $K^*$ is set to $0$ for this period. Thus, we obtain a prepayment option with a lockout provision.

## 4. More Specifications towards Implementation

Given that prepayment options are American call options on defaultable and interest-bearing debts, choosing the right term structure model is crucial in pricing them. We base our valuation on the term-structure dynamics of the HJM to model forward interest rates on defaultable and default-free debt.

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[18] Jarrow (1996) has explained this scheme for a delayed-exercise American call option on a callable coupon bond.

Unlike a general equilibrium model such as Vasicek or CIR, the HJM is an arbitrage-free model that fits the current term structure in the markets. The HJM model uses the initial term structure of interest rates and their volatilities as inputs. By restricting the drift terms of the interest rate to be a function of their volatilities, we can price derivatives under martingales and avoid arbitrage opportunities. These restrictive drifts possess a recursive structure under the “Recovery of Market Value” condition by Duffie and Singleton (1999), which facilitates implementation[19] to a great extent. The HJM is also very flexible to accommodate different structures of interest rate volatilities. This feature is significant as some structures are convenient as they permit closed-form solutions for options and other derivatives. In this section, we first identify the relationship between the drift terms and their volatilities in the spirit of Das and Sundaram (2000). Then we discuss different volatility structures described by Grant and Vora (1999), Chance (2002) and Jarrow (1996). Finally, we present specifications of random variables and their correlation.

## 4.1 Drifts Determination

Drifts $\alpha$ and $\beta$ of the forward-rate and spread processes have to be solved in terms of the two volatilities $\sigma$ and $\eta$. At time $t$, the risk-neutral drift $\alpha$ is

$$\sum_{j=t/\Delta+1}^{T/\Delta-1} \alpha(t, j\Delta) = \frac{1}{\Delta^2} \ln \left\{ E^t \left[ \exp \left( -\Delta^{3/2} \sum_{j=t/\Delta+1}^{T/\Delta-1} \sigma(t, j\Delta) X_1 \right) \right] \right\}, \qquad (0.27)$$

while the risk-neutral drift $\beta$ is

$$\sum_{j=t/\Delta+1}^{T/\Delta-1} \beta(t, j\Delta) = \frac{1}{\Delta^2} \ln \left\{ E^t \left[ \exp \left( -\Delta^{3/2} \sum_{j=t/\Delta+1}^{T/\Delta-1} [\sigma(t, j\Delta) X_1 + \eta(t, j\Delta) X_2] \right) \right] \right\} - \sum_{j=t/\Delta+1}^{T/\Delta-1} \alpha(t, j\Delta). \qquad (0.28)$$

Since $\alpha$ has been solved in terms of $\sigma$, $\beta$ may be solved in terms of $\sigma$ and $\eta$ by replacing $\alpha$ term in the Equation $(0.28)$. We refer to Das and Sundaram (2000) for a detailed derivation of the above results[20].

## 4.2 Volatility Structures

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[19] The HJM produces non-recombining lattice, and the number of nodes in the tree will grow exponentially with the number of steps. However, this recursive procedure can handle these computational problems easily.

[20] For completeness, we include their proof in Appendix 2.

A volatility structure in the one factor HJM model has two dimensions[21]: the cross-sectional and time series volatility. The first dimension refers to the volatility for a set of different forward rates indicated at a given time point, such as $\sigma(t,t)$, $\sigma(t,t+\Delta)$, $\sigma(t,t+2\Delta)$, ..., $\sigma(t,T)$. Time series volatility refers to constant maturity volatility, such as $\sigma(t,T)$, $\sigma(t+\Delta,T)$, $\sigma(t+2\Delta,T)$, ..., $\sigma(T,T)$. In the HJM model, a simple case of volatility structure is in the form of $\sigma(t,T)=\sigma$. For a constant volatility, $\sigma(t,t+\Delta)=\sigma(t+\Delta,t+2\Delta)$, but $\sigma(t,t+\Delta)\neq\sigma(t,t+2\Delta)$. When the volatility of interest is constant, the corresponding interest rate lattice recombines. However, this constant volatility may drive forward interest rates below zero.

Another volatility structure is the nearly proportional volatility, $\sigma(t,T)=\xi(t,T)\min(f(t,T),M)$, where $\xi(t,T)$ is a deterministic function and $M$, as defined before, is a large positive constant. $\sigma(t,T)$ is proportional to the current forward rate and bounds it to the upper end so that it will not be too high.

The third volatility structure is exponentially dampened volatility in the form of $\sigma(t,T)=\sigma e^{-\kappa(T-t)}$. In this specification, volatilities decline at an exponential rate and $\kappa$ is the rate of dampening. Under this time-varying structure, the corresponding interest rate lattice does not recombine because an up-and-down value is not equal to a down-and-up value. This structure is convenient as it yields some closed-form solutions for options and derivatives.

The fourth volatility structure we present is exponentially dampened volatility proportional to the spot rates, specified as $\sigma(t,T-\Delta)=\lambda r(t)^{\gamma}e^{-\kappa(T-t-\Delta)}$. $\lambda$ and $\gamma$ are the proportional factors, $\kappa$ is the rate of dampening, and volatility is proportional to the spot rate. This volatility structure and the nearly proportional volatility structure are stochastic volatilities depending on the level of the interest rates. Like exponentially dampened volatility, this volatility structure results in non-recombining interest rate tree. We will use this volatility structure to illustrate one pricing example in Section five.

## 4.3 Binomial Random Variables and Correlations

To implement the model in a binomial setting, we need to specify the random variables $X_1$, $X_2$ and their correlation. A simple way to binomialize the Wiener process is to let $X_1$ and $X_2$ take a value

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[21] Chance (2002) mentioned three dimensions to refer to the volatility structure of a multi-factor HJM.

of $+1$ and $-1$ at each time step with risk-neutral probabilities of $\frac{1}{2}$. Although it is convenient to set the risk neutral probabilities to $\frac{1}{2}$ to compute the rate and spread non-recombining trees, such as in Das and Sundaram (2000), this is a special property when $\Delta \to 0$. We will show later in the example that risk-neutral probabilities are very close to $\frac{1}{2}$ when $\Delta = 0.25$. Empirical studies have shown that credit spreads and interest rates may be positively or negatively correlated, varying with the business cycle. Assume joint distribution of $(X_1,\, X_2)$ with correlation $\rho$ to be

$$(X_1,\, X_2) = \left\{ \begin{array}{lll} (+1,\,+1)\,, & \text{with probability} & (1+\rho)/4 \\ (+1,\,-1), & \text{with probability} & (1-\rho)/4 \\ (-1,\,+1), & \text{with probability} & (1-\rho)/4 \\ (-1,\,-1), & \text{with probability} & (1+\rho)/4 \end{array} \right\}, \tag{0.29}$$

and we follow this specification in the implementation.

Let $f\,(\cdot)^+$ and $f\,(\cdot)^-$ denote the forward rates that result from $f\,(\cdot)$ if $X_1 = +1$ and $X_1 = -1$; $s\,(\cdot)^+$ and $s\,(\cdot)^-$ the forward spreads from $s\,(\cdot)$ if $X_2 = +1$ and $X_2 = -1$, respectively. We can then compute the state-price $\Lambda$ or $\Gamma$, namely the option price at each node.

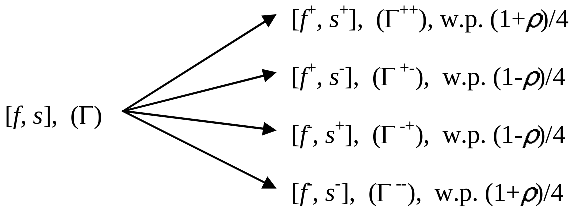

Figure 1: State Price, Probability Distribution and Correlation

Clearly, the state-price $\Gamma(t)$, given not being exercised at time $t$, is the result of the following Equation:

$$\begin{aligned} \Gamma(t) &= E^t\left[\Gamma(t+\Delta)\right]\cdot P(t,t+\Delta) \\ &= [\Gamma(t+\Delta)^{++}\cdot\frac{(1+\rho)}{4} + \Gamma(t+\Delta)^{+-}\cdot\frac{(1-\rho)}{4} \\ &\quad +\Gamma(t+\Delta)^{-+}\cdot\frac{(1-\rho)}{4} + \Gamma(t+\Delta)^{--}\cdot\frac{(1+\rho)}{4}]\cdot P(t,t+\Delta) \end{aligned} \qquad 0.30$$

We are now ready to price the prepayment option in C&I loans.

## 5. A Floating-Rate Loan Example

To demonstrate the model, we price a prepayment option embedded in a floating-rate C&I loan. We model volatility structures as exponentially dampened volatility proportional to the spot rates. This example will take four time steps because a full illustration is impossible with the branches increasing at a rate of $4^j$ when pricing defaultable loans. We follow five steps to compute the option value of a refinancing prepayment. Step 1, we solve the risk-neutral drifts of the forward rate and forward spread processes so as to make the discounted default-free and defaultable debt martingales. Drifts determination should be coupled with the calculation of volatilities as they are proportional to the spot rates/spreads as in subsection 4.1 and 4.2. Step 2, we impose the forward spread non-recombining tree onto the forward rate non-recombining tree, and compute the risky discount factors. Step 3, we compute the market prices of defaultable debt with multiple interest payments to be made; we then repeat Step 1 and 3 to compute the book value of a floating-rate loan, with fixed credit spread at each node. Finally, the prepayment option value is derived through backward induction, assuming that the defaultable discount process governs the behavior of defaultable debt prior to default. Computing the option value of a non-refinancing prepayment follows similar calculations except that the market value of the defaultable debt becomes that of the default-free debt. We then assume an arbitrary probability $\pi$ and calculate the value of prepayment option.

Consider an interest-bearing C&I loan, $L = 100$, $t = 0, T = 1$, $\Delta = 1/4,$ with interest payments occurring at the end of the time step. Interest payments are repriced at the beginning of the time step when the spot interest rate changes. Changes of credit spreads have no influence on the interest payments and we assume the contractual fixed-spread to be $s(0, \frac{3}{4})$. Interest payments $C_1 = 100 \cdot \left[f(0,0) + s(0, \frac{3}{4})\right]/4,$ $C_2 = 100 \cdot \left[f(\frac{1}{4}, \frac{1}{4}) + s(0, \frac{3}{4})\right]/4,$ $C_3 = 100 \cdot \left[f(\frac{1}{2}, \frac{1}{2}) + s(0, \frac{3}{4})\right]/4,$ and $C = 100 \cdot \left[f(\frac{3}{4}, \frac{3}{4}) + s(0, \frac{3}{4})\right]/4$ are scheduled at the end of each node, but indeterministic at time $0$ with the exception of $C_1$. Probability $\pi$ is given as $0.3$; transaction cost is $TC = TC_f + TC_v = 0.04 + 0.06 = 0.1$, fixed throughout all periods; and correlation between $X_1$ and $X_2$ is $-0.08$. The objective is to price the prepayment option without a lockout provision.

We can illustrate this loan's cash flows with Figure 2:

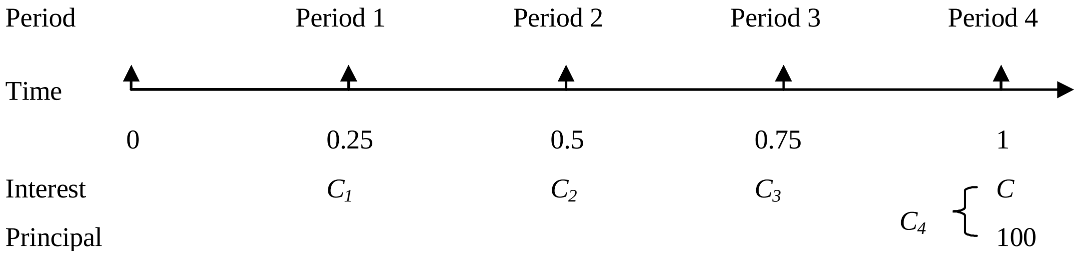

Figure 2: Cash Flows of A Floating-Rate Loan

## 5.1 Interest Rate and Credit Spread non-recombining Trees

The following information is given for the initial term structure of interest rate and credit spread:

| Period | $T$ | $f(0,T-\Delta)$ | $s(0,T-\Delta)$ |
|---|---|---|---|
| 1 | $\frac{1}{4}$ | 0.06 | 0.020 |
| 2 | $\frac{1}{2}$ | 0.07 | 0.021 |
| 3 | $\frac{3}{4}$ | 0.08 | 0.022 |
| 4 | 1 | 0.09 | 0.023 |

(0.31)

The exponentially dampened volatility proportional to the spot rates, $\sigma(t,T-\Delta) = \lambda r(t)^{\gamma}e^{-\kappa(T-t-\Delta)}$, has three parameters to be specified. We have $\lambda = 0.25$, $\gamma = 1$, and $\kappa = 0.05$, which means that the volatility of each forward rate is 25% of the level of the spot rate, dampened by 5% per quarter. We denote $\eta(t,T-\Delta) = \theta r(t)^{\nu}e^{-\delta(T-t-\Delta)}$ as the volatility function for the forward credit spreads. Let $\theta = 0.3, \nu = 1$, and $\delta = 0.04$, then we have the volatility of each forward spread as 30% of the level of the spot spread, dampened by 4% per quarter. By this specification, the spread process entails higher volatility than does the forward rate process.

Interest rate volatilities at time 0 are

| Period | $T$ | $\sigma(t,T-\Delta)$ when $t=0$ |
|---|---|---|
| 1 | $\frac{1}{4}$ | $\sigma(0,0) = 0.25{\times}r(0) \times e^{-0.05*0} = 0.015$ |
| 2 | $\frac{1}{2}$ | $\sigma(0,\frac{1}{4}) = 0.25{\times}r(0) \times e^{-0.05*0.25} = 0.01481367$ |
| 3 | $\frac{3}{4}$ | $\sigma(0,\frac{1}{2}) = 0.25{\times}r(0) \times e^{-0.05*0.50} = 0.01462965$ |
| 4 | 1 | $\sigma(0,\frac{3}{4}) = 0.25{\times}r(0) \times e^{-0.05*0.75} = 0.01444792$ |

(0.32)

and credit spread volatilities at time 0 are

| Period | $T$ | $\eta(t, T-\Delta)$ when $t=0$ |
|---|---|---|
| 1 | $\frac{1}{4}$ | $\eta(0,0) = 0.30\times r(0) \times e^{-0.04*0} = 0.018$ |
| 2 | $\frac{1}{2}$ | $\eta(0,\frac{1}{4}) = 0.30\times r(0) \times e^{-0.04*0.25} = 0.0178209$ |
| 3 | $\frac{3}{4}$ | $\eta(0,\frac{1}{2}) = 0.30\times r(0) \times e^{-0.04*0.50} = 0.0176436$ |
| 4 | 1 | $\eta(0,\frac{3}{4}) = 0.30\times r(0) \times e^{-0.04*0.75} = 0.017468$ |

(0.33)

Recall that drifts are restricted by volatilities with the equation of $\sum_{j=t/\Delta+1}^{T/\Delta-1} \alpha(t,\, j\Delta) = \frac{1}{\Delta^2} \ln \left\{ E^t \left[ \exp \left( -\Delta^{3/2} \sum_{j=t/\Delta+1}^{T/\Delta-1} \sigma(t,\, j\Delta) X_1 \right) \right] \right\}$.

When $T = 1/2$, we have

$$\begin{aligned} \alpha(0, \frac{1}{4}) &= \frac{1}{\Delta^2} \ln \left\{ E^t \left[ \exp \left( -\Delta^{3/2} \sigma(0, \frac{1}{4}) X_1 \right) \right] \right\} \\ &= \ln \left\{ \frac{1}{2} \left[ \exp(-\Delta^{3/2} \sigma(0, \frac{1}{4})) + \exp \left( \Delta^{3/2} \sigma(0, \frac{1}{4}) \right) \right] \right\} / \Delta^2 \\ &= 0.000027431. \end{aligned} \qquad 0.34$$

When $T = 3/4$, we have

$$\begin{aligned} \alpha(0, \frac{1}{4}) + \alpha(0, \frac{1}{2}) &= \frac{1}{\Delta^2} \ln \left\{ E^t \left[ \exp \left( -\Delta^{3/2} \left[ \sigma(0, \frac{1}{4}) + \sigma(0, \frac{1}{2}) \right] X_1 \right) \right] \right\} \\ &= \ln \left\{ \frac{1}{2} \left[ \begin{array}{c} \exp(-\Delta^{3/2} \left[ \sigma(0, \frac{1}{4}) + \sigma(0, \frac{1}{2}) \right]) \\ + \exp \left( \Delta^{3/2} \left[ \sigma(0, \frac{1}{4}) + \sigma(0, \frac{1}{2}) \right] \right) \end{array} \right] \right\} / \Delta^2 \\ &= 0.000108363. \end{aligned} \qquad 0.35$$

We can calculate $\alpha(0, \frac{3}{4})$ similarly. The three drift terms at time 0 are

$\alpha(0, \frac{1}{4}) = 0.000027431,$

$\alpha(0, \frac{1}{2}) = 0.000080933,$

$\alpha(0, \frac{3}{4}) = 0.000132440.$

These calculations enable us to fill the first time step of the interest rate tree:

$$\begin{bmatrix} f(0,\frac{3}{4}) = 0.09 \\ f(0,\frac{1}{2}) = 0.08 \\ f(0,\frac{1}{4}) = 0.07 \\ f(0,0) = 0.06 \end{bmatrix} \quad \begin{array}{l} \begin{bmatrix} f(\frac{1}{4},\frac{3}{4})^{+}{=}f(0,\frac{3}{4}) + \alpha(0,\frac{3}{4})0.25 + \sigma(0,\frac{3}{4})\sqrt{0.25}{=}\ 0.097257068 \\ f(\frac{1}{4},\frac{1}{2})^{+}{=}f(0,\frac{1}{2}) + \alpha(0,\frac{1}{2})0.25 + \sigma(0,\frac{1}{2})\sqrt{0.25}{=}\ 0.087335058 \\ f(\frac{1}{4},\frac{1}{4})^{+}{=}f(0,\frac{1}{4}) + \alpha(0,\frac{1}{4})0.25 + \sigma(0,\frac{1}{4})\sqrt{0.25}{=}\ 0.077413691 \end{bmatrix} \\ \text{with equal probability of 0.5} \\ \begin{bmatrix} f(\frac{1}{4},\frac{3}{4})^{-}{=}f(0,\frac{3}{4}) + \alpha(0,\frac{3}{4})0.25 - \sigma(0,\frac{3}{4})\sqrt{0.25}{=}\ 0.082809152 \\ f(\frac{1}{4},\frac{1}{2})^{-}{=}f(0,\frac{1}{2}) + \alpha(0,\frac{1}{2})0.25 - \sigma(0,\frac{1}{2})\sqrt{0.25}{=}\ 0.072705409 \\ f(\frac{1}{4},\frac{1}{4})^{-}{=}f(0,\frac{1}{4}) + \alpha(0,\frac{1}{4})0.25 - \sigma(0,\frac{1}{4})\sqrt{0.25}{=}\ 0.062600024 \end{bmatrix} \end{array} \tag{0.36}$$

At time 0.25, the volatilities depend on the realized spot rate $f(\frac{1}{4},\frac{1}{4})$, we consider the upper branch of the tree:

| Period | $T$ | $\sigma(\frac{1}{4}, T-\Delta)$ |
|---|---|---|
| 2 | $\frac{1}{2}$ | $\sigma(\frac{1}{4},\frac{1}{4}) = 0.25 \times f(\frac{1}{4},\frac{1}{4})^{+} \times e^{-0.05*0} =$ 0.019353423 |
| 3 | $\frac{3}{4}$ | $\sigma(\frac{1}{4},\frac{1}{2}) = 0.25 \times f(\frac{1}{4},\frac{1}{4})^{+} \times e^{-0.05*0.25} =$ 0.019113011 |
| 4 | 1 | $\sigma(\frac{1}{4},\frac{3}{4}) = 0.25 \times f(\frac{1}{4},\frac{1}{4})^{+} \times e^{-0.05*0.50} =$ 0.018875585 |

(0.37)

We use these volatilities to compute the two drift terms $\alpha(\frac{1}{4},\frac{1}{2})$ and $\alpha(\frac{1}{4},\frac{3}{4})$ as

$\alpha(\frac{1}{4},\frac{1}{2}) =$ 0.0000456634,

$\alpha(\frac{1}{4},\frac{3}{4}) =$ 0.000134728.

From the drift terms, volatilities, and forward rates at time 0.25, we can derive the non-recombining tree at time 0.5.

$$\begin{bmatrix} f(\frac{1}{4},\frac{3}{4}) = 0.097257068 \\ f(\frac{1}{4},\frac{1}{2}) = 0.087335058 \\ f(\frac{1}{4},\frac{1}{4}) = 0.077413691 \end{bmatrix} \quad \begin{array}{l} \begin{bmatrix} f(\frac{1}{2},\frac{3}{4})^{+}{=}f(\frac{1}{4},\frac{3}{4}) + \alpha(\frac{1}{4},\frac{3}{4})0.25 + \sigma(\frac{1}{4},\frac{3}{4})\sqrt{0.25}{=}\ 0.106729 \\ f(\frac{1}{2},\frac{1}{2})^{+}{=}f(\frac{1}{4},\frac{1}{2}) + \alpha(\frac{1}{4},\frac{1}{2})0.25 + \sigma(\frac{1}{4},\frac{1}{2})\sqrt{0.25}{=}\ 0.096903 \end{bmatrix} \\ \text{with equal probability of 0.5} \\ \begin{bmatrix} f(\frac{1}{2},\frac{3}{4})^{-}{=}f(\frac{1}{4},\frac{3}{4}) + \alpha(\frac{1}{4},\frac{3}{4})0.25 - \sigma(\frac{1}{4},\frac{3}{4})\sqrt{0.25}{=}\ 0.087853 \\ f(\frac{1}{2},\frac{1}{2})^{-}{=}f(\frac{1}{4},\frac{1}{2}) + \alpha(\frac{1}{4},\frac{1}{2})0.25 - \sigma(\frac{1}{4},\frac{1}{2})\sqrt{0.25}{=}\ 0.077790 \end{bmatrix} \end{array} \tag{0.38}$$

Proceeding in this way, we then produce an entire interest rate non-recombining tree as shown below:

$$\begin{tikzcd}[row sep=tiny]
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.118709\right] \\
 & & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4})=0.106729 \\ f(\frac{1}{2},\frac{1}{2})=0.096903 \end{bmatrix} & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.094784\right] \\
 & \begin{bmatrix} f(\frac{1}{4},\frac{3}{4})=0.0972571 \\ f(\frac{1}{4},\frac{1}{2})=0.0873351 \\ f(\frac{1}{4},\frac{1}{4})=0.0774137 \end{bmatrix} \arrow[uur] \arrow[ddr] & & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.0974674\right] \\
 & & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4})=0.087853 \\ f(\frac{1}{2},\frac{1}{2})=0.077790 \end{bmatrix} & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.0782615\right] \\
\begin{bmatrix} f(0,\frac{3}{4})=0.09 \\ f(0,\frac{1}{2})=0.08 \\ f(0,\frac{1}{4})=0.07 \\ f(0,0)=0.06 \end{bmatrix} \arrow[uuuur] \arrow[ddddr] & \text{with equal probability of 0.5} & & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.100405\right] \\
 & & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4})=0.090463 \\ f(\frac{1}{2},\frac{1}{2})=0.080440 \end{bmatrix} & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.0805451\right] \\
 & \begin{bmatrix} f(\frac{1}{4},\frac{3}{4})=0.0828092 \\ f(\frac{1}{4},\frac{1}{2})=0.0727054 \\ f(\frac{1}{4},\frac{1}{4})=0.0626000 \end{bmatrix} \arrow[uur] \arrow[ddr] & & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.0832296\right] \\
 & & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4})=0.0751994 \\ f(\frac{1}{2},\frac{1}{2})=0.0649851 \end{bmatrix} & \\
 & & & \left[f(\frac{3}{4},\frac{3}{4})=0.0671852\right]
\end{tikzcd} \tag{0.39}$$

We now come to check if the assumed risk-neutral probability is really 0.5. This system is arbitrage-free if and only if there exist unique risk-neutral probabilities $\phi(t,T)^+$ such that

$$\phi(t,T)^+ = \frac{f(t+\Delta,T)^+ - f(t,T)}{f(t+\Delta,T)^+ - f(t+\Delta,T)^-}, \tag{0.40}$$

and the downward probabilities are $\phi(t,T)^-$ such that

$$\phi(t,T)^- = \frac{f(t,T) - f(t+\Delta,T)^-}{f(t+\Delta,T)^+ - f(t+\Delta,T)^-}. \tag{0.41}$$

Verify these probabilities with node $f(\frac{1}{2},\frac{1}{2})$ relative to $f(\frac{1}{4},\frac{1}{2})$, we have:

$$\begin{aligned} \phi(\frac{1}{4},\frac{1}{2})^+ &= \frac{f(\frac{1}{2},\frac{1}{2})^+ - f(\frac{1}{4},\frac{1}{2})}{f(\frac{1}{2},\frac{1}{2})^+ - f(\frac{1}{2},\frac{1}{2})^-} \\ &= \frac{0.096902979 - 0.08733506}{0.096902979 - 0.077789968} \\ &= 0.500597281. \end{aligned} \tag*{0.42}$$

The smaller $\Delta$ becomes, the closer $\phi(t,T)$ approaches 0.5.

For a defaultable floating-rate contract, interest payments reset every period according to the spot rate plus a fixed credit spread. In this example, we recall that we have assumed that the fixed portion is equivalent to $s(0, \frac{3}{4})$, observed at time 0. At the end of each period, the interest payment owed $C_j$ is paid. At the end of the contract, $C_{T/\Delta}$, which includes the interest $C$ plus principal $L$, is paid. The floating interest is depicted with the following tree:

| Time 0 | Time $\frac{1}{4}, C_1$ | Time $\frac{1}{2}, C_2$ | Time $\frac{3}{4}, C_3$ | Time $1, C_4$ |
|---|---|---|---|---|
| | | | | 103.542720847 |
| | | | 2.997574468 | |
| | | | | 102.944600656 |
| | | 2.510342279 | | |
| | | | | 103.01168601 |
| | | | 2.519749201 | |
| | | | | 102.531538225 |
| 0 | $C_1 = 100 \cdot \left[ f(0,0) + s(0, \frac{3}{4}) \right] /4 = 2.075$ | | | |
| | | | | 103.085137091 |
| | | | 2.586016825 | |
| | | | | 102.588628198 |
| | | 2.140000604 | | |
| | | | | 102.655741134 |
| | | | 2.199626861 | |
| | | | | 102.254629778 |
| $E^0$ | $E^0[C_1] = 2.075$ | $E^0[C_2]$ =2.33 | $E^0[C_3]$ =2.57 | $E^0[C_4]$ =102.8 |

(0.43)

At time 0, we are only certain about the interest owed at the end of the first period. The other interest to be paid follows the evolution of the entire risk-free floating rate curve. Because we know the distribution of the underlying stochastic process at time 0, interest payments can be simultaneously determined as an average of the future possible cash flows[22]. Interest payments are definitely path dependent as time will reveal more information on the development of the forward rate curve, so will change the expected value of future cash flows.

Similar to the forward rate non-recombining tree example, we can produce an entire non-recombining tree for forward spread according to Equation$(0.28)$.

---

[22] By average, we assume 0.5 risk-neutral probability for both up and down node.

$$
\begin{array}{llll}
 & & & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.118709\right] \\
 & & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4}) = 0.106729 \\ f(\frac{1}{2},\frac{1}{2}) = 0.096903 \end{bmatrix} & \\
 & & \nearrow & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.094784\right] \\
 & \begin{bmatrix} f(\frac{1}{4},\frac{3}{4}) = 0.0972571 \\ f(\frac{1}{4},\frac{1}{2}) = 0.0873351 \\ f(\frac{1}{4},\frac{1}{4}) = 0.0774137 \end{bmatrix} & & \\
 & & \searrow & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.0974674\right] \\
 & \nearrow & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4}) = 0.087853 \\ f(\frac{1}{2},\frac{1}{2}) = 0.077790 \end{bmatrix} & \\
 & & & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.0782615\right] \\
\begin{bmatrix} f(0,\frac{3}{4}) = 0.09 \\ f(0,\frac{1}{2}) = 0.08 \\ f(0,\frac{1}{4}) = 0.07 \\ f(0,0) = 0.06 \end{bmatrix} & \text{with equal probability of 0.5} & & \\
 & & & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.100405\right] \\
 & \searrow & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4}) = 0.090463 \\ f(\frac{1}{2},\frac{1}{2}) = 0.080440 \end{bmatrix} & \\
 & & \nearrow & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.0805451\right] \\
 & \begin{bmatrix} f(\frac{1}{4},\frac{3}{4}) = 0.0828092 \\ f(\frac{1}{4},\frac{1}{2}) = 0.0727054 \\ f(\frac{1}{4},\frac{1}{4}) = 0.0626000 \end{bmatrix} & & \\
 & & \searrow & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.0832296\right] \\
 & & \begin{bmatrix} f(\frac{1}{2},\frac{3}{4}) = 0.0751994 \\ f(\frac{1}{2},\frac{1}{2}) = 0.0649851 \end{bmatrix} & \\
 & & & \left[f(\tfrac{3}{4},\tfrac{3}{4}) = 0.0671852\right]
\end{array}
\tag{0.44}
$$

Credit spread itself is a function of forward interest rate, interest rate volatility and credit spread volatility, with credit spread volatility being higher than that of the interest rate. As a consequence, credit spread calculated might go below zero. We then reset the negative spread at node $s(\frac{3}{4},\frac{3}{4})^{---}$ to $0$. Next step involves imposing the spread tree onto the forward rate tree to create the dynamics of risky forward rates.

## 5.2 Double non-recombining Tree as Discount Factor

The double non-recombining tree increases its nodes at a speed of $4^j$. To illustrate how the model works, we just show the dynamics to the second period. Defaultable forward interest rate $\varphi(t,T)$ equals $f(t,T)+s(t,T)$, with four possible development in the following period: $\varphi(\cdot)^{++} = f(\cdot)^{+} + s(\cdot)^{+}$, $\varphi(\cdot)^{+-} = f(\cdot)^{+} + s(\cdot)^{-}$, $\varphi(\cdot)^{-+} = f(\cdot)^{-} + s(\cdot)^{+}$, and $\varphi(\cdot)^{--} = f(\cdot)^{-} + s(\cdot)^{-}$. The four nodes developed from the original rates are:

$$
\begin{array}{cl}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \\
 & \begin{bmatrix} \varphi(\frac{1}{4},\frac{3}{4})^{++} = f(\frac{1}{4},\frac{3}{4})^{+} + s(\frac{1}{4},\frac{3}{4})^{+} = 0.129030668 \\ \varphi(\frac{1}{4},\frac{1}{2})^{++} = f(\frac{1}{4},\frac{1}{2})^{+} + s(\frac{1}{4},\frac{1}{2})^{+} = 0.118181658 \\ \varphi(\frac{1}{4},\frac{1}{4})^{++} = f(\frac{1}{4},\frac{1}{4})^{+} + s(\frac{1}{4},\frac{1}{4})^{+} = 0.107332791 \end{bmatrix} \cdots \\
 & \begin{bmatrix} \varphi(\frac{1}{4},\frac{3}{4})^{+-} = f(\frac{1}{4},\frac{3}{4})^{+} + s(\frac{1}{4},\frac{3}{4})^{-} = 0.111562568 \\ \varphi(\frac{1}{4},\frac{1}{2})^{+-} = f(\frac{1}{4},\frac{1}{2})^{+} + s(\frac{1}{4},\frac{1}{2})^{-} = 0.100538058 \\ \varphi(\frac{1}{4},\frac{1}{4})^{+-} = f(\frac{1}{4},\frac{1}{4})^{+} + s(\frac{1}{4},\frac{1}{4})^{-} = 0.089511891 \end{bmatrix} \cdots \\
\begin{bmatrix} \varphi(0,\frac{3}{4}) = f(0,\frac{3}{4}) + s(0,\frac{3}{4}) = 0.113 \\ \varphi(0,\frac{1}{2}) = f(0,\frac{1}{2}) + s(0,\frac{1}{2}) = 0.102 \\ \varphi(0,\frac{1}{4}) = f(0,\frac{1}{4}) + s(0,\frac{1}{4}) = 0.091 \\ \varphi(0,0) = f(0,0) + s(0,0) = 0.080 \end{bmatrix} & \\
 & \begin{bmatrix} \varphi(\frac{1}{4},\frac{3}{4})^{-+} = f(\frac{1}{4},\frac{3}{4})^{-} + s(\frac{1}{4},\frac{3}{4})^{+} = 0.114582752 \\ \varphi(\frac{1}{4},\frac{1}{2})^{-+} = f(\frac{1}{4},\frac{1}{2})^{-} + s(\frac{1}{4},\frac{1}{2})^{+} = 0.103552009 \\ \varphi(\frac{1}{4},\frac{1}{4})^{-+} = f(\frac{1}{4},\frac{1}{4})^{-} + s(\frac{1}{4},\frac{1}{4})^{+} = 0.092519124 \end{bmatrix} \cdots \\
 & \begin{bmatrix} \varphi(\frac{1}{4},\frac{3}{4})^{--} = f(\frac{1}{4},\frac{3}{4})^{-} + s(\frac{1}{4},\frac{3}{4})^{-} = 0.097114652 \\ \varphi(\frac{1}{4},\frac{1}{2})^{--} = f(\frac{1}{4},\frac{1}{2})^{-} + s(\frac{1}{4},\frac{1}{2})^{-} = 0.085908409 \\ \varphi(\frac{1}{4},\frac{1}{4})^{--} = f(\frac{1}{4},\frac{1}{4})^{-} + s(\frac{1}{4},\frac{1}{4})^{-} = 0.074698224 \end{bmatrix} \cdots \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots
\end{array}
\tag{0.45}
$$

## 5.3 Value of Refinancing Prepayment Option

Before valuing the refinancing prepayment option, we need to value the market price of an interest-bearing defaultable loan, which has to be preceded with the pricing of a zero-interest defaultable loan. A zero-interest defaultable loan is a function of the risky forward interest rate: $P^*(t,T) = \exp\left\{-\sum_{j=t/\Delta}^{T/\Delta-1}\varphi(t,j\Delta)\cdot\Delta\right\}$. For example, $P^*(0,\frac{1}{4}) = \exp\left\{-\varphi(0,0)\cdot 0.25\right\} = 0.980198673$; $P^*(0,\frac{1}{2}) = \exp\left\{-[\varphi(0,0)+\varphi(0,\frac{1}{4})]\cdot 0.25\right\} = 0.958150898$. We can then build a tree for this defaultable zero-interest loan. The following tree is a two-period illustration:

$$\begin{array}{ll}
 & \qquad\qquad\qquad\qquad\cdots \\
 & \begin{bmatrix} P^*(\frac{1}{4},1)^{++}= 0.915178382 \\ P^*(\frac{1}{4},\frac{3}{4})^{++}= 0.945181211 \\ P^*(\frac{1}{4},\frac{1}{2})^{++}= 0.973523614 \end{bmatrix} \\
 & \begin{bmatrix} P^*(\frac{1}{4},1)^{+-}= 0.927369561 \\ P^*(\frac{1}{4},\frac{3}{4})^{+-}= 0.953598565 \\ P^*(\frac{1}{4},\frac{1}{2})^{+-}= 0.977870557 \end{bmatrix} \cdots \\
\begin{bmatrix} P^*(0,1) = 0.908009898 \\ P^*(0,\frac{3}{4}) = 0.934026938 \\ P^*(0,\frac{1}{2}) = 0.958150898 \\ P^*(0,\frac{1}{4}) = 0.980198673 \end{bmatrix} & \\
 & \begin{bmatrix} P^*(\frac{1}{4},1)^{-+}= 0.925275756 \\ P^*(\frac{1}{4},\frac{3}{4})^{-+}= 0.952164197 \\ P^*(\frac{1}{4},\frac{1}{2})^{-+}= 0.977135662 \end{bmatrix} \cdots \\
 & \begin{bmatrix} P^*(\frac{1}{4},1)^{--}= 0.937601443 \\ P^*(\frac{1}{4},\frac{3}{4})^{--}= 0.960643739 \\ P^*(\frac{1}{4},\frac{1}{2})^{--}= 0.981498733 \end{bmatrix} \cdots \\
 & \qquad\qquad\qquad\qquad\cdots
\end{array} \tag{0.46}$$

The market value of an interest-bearing defaultable loan is the sum of present value of the interest and principal paid. We have its pricing formula as $D^*(t) = \sum_{j=t/\Delta+1}^{T/\Delta} E^t\left[C_j\right] \cdot P^*(t, j\Delta)$; $C_j$ is considered with embedded credit risk, discounted by risky discount factor $P^*$. Note that we only know the interest owed for the current period, and future interest payments are expected value with respect to different probabilities. The market value of this interest-bearing defaultable loan at time 0 is

$$\begin{aligned}
D^*(0) &= \sum_{j=1}^{4} E^0\left[C_j\right] P^*(0, j\Delta) \\
&= C_1 P^*(0, \frac{1}{4}) + E^0\left[C_2\right] P^*(0, \frac{1}{2}) + E^0\left[C_3\right] P^*(0, \frac{3}{4}) + E^0\left[C_4\right] P^*(0, 1) \\
&= 100.03.
\end{aligned} \tag*{0.47}$$

Jarrow (1996) has proved that for a default-free floating-rate loan, as the interest payment owed is paid at the end of each period, the outstanding borrowing resets to the principal. The idea works in the same way for a defaultable loan that if we add on a credit spread, the interest owed is paid and the value of the cash flows from the floating-rate loan equals the amount borrowed. $D^*(0)$ is not exactly equal to 100 due to the approximate nature of this example. Recall that approximation may drive risk-neutral probabilities departing from 0.5 slightly. The analysis emphasizes the market value of this loan when credit spread changes. Changing credit spread works directly on the market value of the zero-interest loan, the discount factor, therefore, the market value of this interest-bearing defaultable loan fluctuates accordingly. In theory, the book value of this loan after interest equals the amount borrowed. In the numerical example, the book value of the loan is lower than 100 for the fixed credit spread $s(0, \frac{3}{4})$ which we assumed is higher than the spot spread $s(0,0)$[23]. Using the formula for pricing $D^*(t)$, we have the tree for the market value of $D^*(t)$ with changing credit spread. For the book value of the loan, $N^*(t,T)$ should be first derived as with $P^*(t,T)$, and $V^*(t)$ follows the same procedure as with calculating $D^*(t)$. The option value for the first period without considering further branches can be depicted as follows:

$$\begin{array}{ll}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \\
 & \left[\begin{array}{c} \left[\begin{array}{c} D^*(\frac{1}{4}) = 99.32 \\ V^*(\frac{1}{4}) = 99.89 \end{array}\right] \Longrightarrow \\ \\ \Gamma(\frac{1}{4})^{++} = \max\left[D^*(\frac{1}{4}) - V^*(\frac{1}{4}) - TC(\frac{1}{4}), 0\right] = \qquad 0 \end{array}\right] \\
 & \left[\begin{array}{c} \left[\begin{array}{c} D^*(\frac{1}{4}) = 100.6102 \\ V^*(\frac{1}{4}) = 99.8906 \end{array}\right] \Longrightarrow \\ \\ \Gamma(\frac{1}{4})^{+-} = \max\left[D^*(\frac{1}{4}) - V^*(\frac{1}{4}) - TC(\frac{1}{4}), 0\right] = 0.6196 \end{array}\right] \\
\left[\begin{array}{c} \left[\begin{array}{c} D^*(0) = 100.03 \\ V^*(0) = 99.89 \end{array}\right] \Longrightarrow \\ \\ D^*(0) - V^*(0) - TC(0) = 0.04 \end{array}\right] & \\
 & \left[\begin{array}{c} \left[\begin{array}{c} D^*(\frac{1}{4}) = 99.344 \\ V^*(\frac{1}{4}) = 99.917 \end{array}\right] \Longrightarrow \\ \\ \Gamma(\frac{1}{4})^{-+} = \max\left[D^*(\frac{1}{4}) - V^*(\frac{1}{4}) - TC(\frac{1}{4}), 0\right] = \qquad 0 \end{array}\right] \\
 & \left[\begin{array}{c} \left[\begin{array}{c} D^*(\frac{1}{4}) = 100.639 \\ V^*(\frac{1}{4}) = 99.917 \end{array}\right] \Longrightarrow \\ \\ \Gamma(\frac{1}{4})^{--} = \max\left[D^*(\frac{1}{4}) - V^*(\frac{1}{4}) - TC(\frac{1}{4}), 0\right] = 0.6222 \end{array}\right] \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots
\end{array} \tag{0.48}$$

Recall that the value of a refinancing prepayment option $\Gamma(t)$ :

$$\Gamma(t) = \max\left\{E^t\left[\Gamma(t+\Delta)\right] \cdot P(t, t+\Delta),\ D^*(t) - V^*(t) - TC(t)\right\}. \tag{0.49}$$

---

[23] A direct result of this assumption is that the loan will be discounted at a higher rate.

At time $0$, we have

$$\begin{aligned}\Gamma(0) &= \max\left\{E^0\left[\Gamma(\frac{1}{4})\right]\cdot P(0,\frac{1}{4}),\, D^*(0)-V^*(0)-TC(0)\right\}\\ &= \max\left\{\left[\begin{array}{c}\frac{(1+\rho)}{4}\Gamma(\frac{1}{4})^{++}+\frac{(1-\rho)}{4}\Gamma(\frac{1}{4})^{+-}\\ +\frac{(1-\rho)}{4}\Gamma(\frac{1}{4})^{-+}+\frac{(1+\rho)}{4}\Gamma(\frac{1}{4})^{--}\end{array}\right]\cdot P(0,\frac{1}{4}),\, 0.04\right\}\\ &= 0.306.\end{aligned} \tag{0.50}$$

We assume a non-refinancing prepayment probability of 0.3. Consequently, the solution for the value of a refinancing prepayment option is $0.214$ at time $0$. The prepayment option value decreases when approaching maturity; calculations for further periods have shown that it is optimal to exercise the option at the beginning of the contract. Therefore, a lockout provision that prohibits prepayment at the beginning of the contract will reduce the option value significantly.

## 5.4 Value of Prepayment Option and Its Sensitivity

The option value of the non-refinancing prepayment can be derived with the exact procedure, except that the market value of a defaultable debt becomes that of a default-free debt. We suppress the demonstration to make the paper reasonably concise. Non-refinancing prepayment option is more valuable to reach $1.67229$ with all the information given in the example. The arbitrary probability $\pi$ is $0.3$, which rescales the value of the non-refinancing prepayment option to $0.502$. In summary, the value of this prepayment option overall at time $0$ is

$$\begin{aligned}A(0) &= \pi\cdot\Lambda(0)+(1-\pi)\cdot\Gamma(0)\\ &= 0.3\cdot 1.67229+0.7\cdot 0.306\\ &= 0.716.\end{aligned} \tag{0.51}$$

We will consider variations from this base case. Changing $\pi$ from $0$ to $1$ will increase the prepayment option value from $0.306$ to $1.67229$. The higher the probability of the non-refinancing prepayment is, the higher the value of the prepayment option becomes. Holding all else the same, additional comparative statics show that:

– Lockout provisions reduce the prepayment option value. For example, if the contract prohibits prepayment during the first period, the prepayment option value is reduced to $0.4925$; no prepayment during the first two periods would reduce the option value to $0.236$.

– Increasing transaction costs to $0.2$ decreases the value of the refinancing prepayment option to $0.26$, and the prepayment option value as a whole is reduced to $0.684$.

– Increasing correlation of the forward interest rate and forward spread to $0.08$, increases the value of the refinancing prepayment option to $0.31$. But it brings no impact on the value of the non-refinancing prepayment option.

– Increasing the contract term from 1 to 2 years, with four repricing periods remaining the same increases the value of the refinancing prepayment option to $0.85$ and the value of the non-refinancing prepayment option to $3.08$. The overall prepayment option value would have been increased to $1.52$.

– The prepayment option value is insensitive to the forward interest rate volatility which is consistent with Duffie and Singleton (1999). Increasing $\lambda$ to $0.5$, doubling volatility of the forward interest rate, increases the value of the refinancing prepayment option to $0.31$, not much from the base case $0.306$. An increase of $\lambda$ leaves the value of the non-refinancing prepayment option unchanged.

– But the prepayment option value is sensitive to the forward credit spread volatility. Increasing $\theta$ to $0.6$ increases the value of the refinancing prepayment option to $0.6336$, with no impact on the non-refinancing prepayment option value. The overall option value increases to $0.945$.

– Increasing the initial forward interest rate curve increases the value of the prepayment option overall, consistent with Dunn and Spatt (1999). Note that when increasing the initial forward interest rate to $\{0.08, 0.09, 0.10, 0.11\}$, the value of the refinancing prepayment option increases to $0.416$, but the value of the non-refinancing prepayment option is reduced to $1.655$. The overall prepayment option value increases to $0.788$. In contrast, when decreasing the initial rate curve to $\{0.05, 0.06, 0.07, 0.08\}$, the value of the refinancing prepayment option decreases to $0.257$, and the value of the non-refinancing prepayment option increases to $1.68$. The overall prepayment option value due to the lower initial forward rate curve is reduced to $0.68$. It seems that the value of the refinancing prepayment option is more sensitive to the initial forward interest rate than is that of the non-refinancing prepayment option.

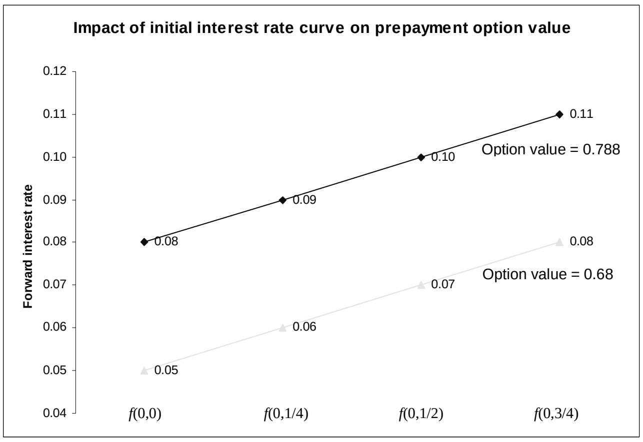

Figure 3: Impact of Initial Interest Rate Curve on Prepayment Option Value

– An increasing initial credit spread curve leads to a higher option value and a decreasing spread curve maintains a lower prepayment option value. Increasing initial credit spread to $\{0.02, 0.022, 0.024, 0.026\}$ increases the value of the refinancing option to $0.3476$, non-refinancing option to $1.89$, and the overall prepayment option value reaches $0.81$. Decreasing initial credit spread to a humped-shape $\{0.02, 0.024, 0.022, 0.018\}$ decreases the value of the refinancing option to $0.206$, non-refinancing option to $1.308$, and the overall prepayment option value becomes $0.5366$. Both classes of prepayment are sensitive to the forward credit spread curve.

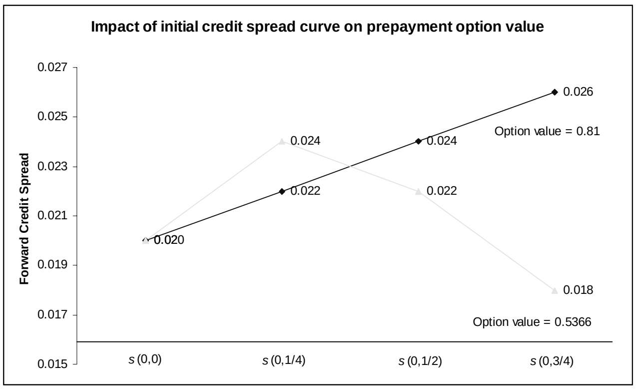

Figure 4: Impact of Initial Credit Spread Curve on Prepayment Option Value

# 6. Conclusion

We have developed a discrete time model for pricing prepayment option in C&I loans at origination. The model deals with different volatility structures, multiple interest payments, lockout provisions, and transaction costs and fees with a double non-recombining tree procedure. Prepayment in C&I loans follows a rational decision process which may be modeled cleanly with a distinction between refinancing and non-refinancing prepayment. The firm specific credit risk has been priced through a credit spread process. Besides that, the method can be modified to value many credit derivatives.

The model can be improved in many ways. The discrete time implementation involves approximation which may be enhanced with a truly arbitrage-free model. Further extension of the model includes empirically specified prepayment probability; a hazard model might describe prepayment probability well. The model is the first in its kind to price the prepayment option in C&I loans fairly and this is the only way to make prepayment a boon rather than a burden.

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# Appendix 1: Valuation of Defaultable Debt

Consider a firm issuing defaultable debt. $P^*(t,T)$ represents the time $t$ market price of the debt, which is risky because if the firm defaults prior to time $T$, the promised payment may not be made. As in Duffie and Singleton (1999), let $h_t$ denote default hazard rate at time $t$, and $l_t$ denote expected fractional loss in market value. So $h_t\Delta$ gives the approximate probability of default over the time interval $(t,t+\Delta)$. The recovery in the event of default at time $t$ is $\Phi_t$ and the risk-free short rate is $r(t)$. The proof is to show that $P^*(t,T)$ may be priced like a default-free debt by replacing risk-free interest rate process with a risk-adjusted process.

Under the assumption of no arbitrage and complete markets, there exists a unique risk-neutral probability $Q$ such that the present value of the debt is computed by discounting at the risk-free short rate and then taking an expectation with respect to $Q$. If default has not occurred by time $t$, $P^*(t,T)$ would be the present value of $\Phi_{t+\Delta}$ if default occurs between $t$ and $t+\Delta$, plus the present value of receiving $P^*(t+\Delta,T)$ in the event of no default. That is

$$P^*(t,T) = \exp\left\{-r(t)\Delta\right\} h_t\Delta E^t\left[\Phi_{t+\Delta}\right] + \exp\left\{-r(t)\Delta\right\}(1-h_t\Delta)E^t\left[P^*(t+\Delta,T)\right]. \qquad (0.52)$$

If the recovery is taken to be a fraction of the market value at time $t+\Delta$, termed as “Recovery of Market Value” (RMV), we then have $E^t\left[\Phi_{t+\Delta}\right] = (1-l_t)E^t\left[P^*(t+\Delta,T)\right]$. Substituting RMV into Equation (0.52) gives

$$\begin{aligned} P^*(t,T) &= \exp\left\{-r(t)\Delta\right\} h_t\Delta(1-l_t)E^t\left[P^*(t+\Delta,T)\right] \\ &\quad + \exp\left\{-r(t)\Delta\right\}(1-h_t\Delta)E^t\left[P^*(t+\Delta,T)\right] \\ &= \exp\left\{-r(t)\Delta\right\}(1-h_tl_t\Delta)E^t\left[P^*(t+\Delta,T)\right]. \qquad 0.53 \end{aligned}$$

Using the approximation of $\exp\{c\}=1+c$ for small $c$, we have

$$\exp\left\{-h_tl_t\Delta\right\} \simeq 1-h_tl_t\Delta. \qquad (0.54)$$

Thus, $P^*(t,T)$ may be discounted at a risk-adjusted rate $[r(t)+h_tl_t]$ for the time interval $(t,t+\Delta)$,

$$P^*(t,T) = \exp\left\{-\left[r(t)+h_tl_t\right]\cdot\Delta\right\} E^t\left[P^*(t+\Delta,T)\right]. \qquad (0.55)$$

We therefore define $h_t l_t$ as short credit spread $s(t,t)$. Solving Equation (0.55) forward recursively over the life of the debt,

$$\begin{aligned}
P^*(t,T) &= \exp\left\{-\left[r(t)+s(t,t)\right]\cdot\Delta\right\}E^t\left[P^*(t+\Delta,T)\right] \\
&= \exp\left\{-\left[r(t)+s(t,t)\right]\cdot\Delta\right\}\cdot E^t\left[\exp\left\{-\left[\begin{array}{c} f(t+\Delta,t+\Delta) \\ +s(t+\Delta,t+\Delta)\end{array}\right]\cdot\Delta\right\}\right] \\
&\quad \cdot E^t\left[P^*(t+2\Delta,T)\right]\cdot\ldots\cdot E^t\left[P^*(T,T)\right] \\
&= \exp\left\{-\left[r(t)+s(t,t)\right]\cdot\Delta\right\}\cdot\exp\left\{-\left[f(t,t+\Delta)+s(t,t+\Delta)\right]\cdot\Delta\right\} \\
&\quad \cdot E^t\left[P^*(t+2\Delta,T)\right]\cdot\ldots\cdot 1 \\
&= \exp\left\{-\sum_{j=t/\Delta}^{T/\Delta-1}\left[f(t,j\Delta)+s(t,j\Delta)\right]\cdot\Delta\right\} \\
&= \exp\left\{-\sum_{j=t/\Delta}^{T/\Delta-1}\varphi(t,j\Delta)\cdot\Delta\right\} \qquad 0.56
\end{aligned}$$

Thus, the market price of a defaultable debt can be viewed as if it were default-free, discounted by the risky forward rate $\varphi(t,j\Delta)$.

# Appendix 2: Arbitrage-free Drifts Determination

We first solve drift $\alpha$ in terms of $\sigma$. Define $B(t)$ as the time $t$ value of a "money-money account" that uses an initial investment of \$1, so

$$B(t) = \exp\left\{ \sum_{j=0}^{t/\Delta-1} r(j\Delta)\cdot\Delta \right\}. \tag{0.57}$$

Under $Q$ measure, all debt prices discounted by $B(t)$ are martingales. Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$ :

$$Z(t,T) = \frac{P(t,T)}{B(t)}. \tag{0.58}$$

As $Z$ is a martingale under $Q$, we have

$$\begin{aligned}
E^t\left[\frac{Z(t+\Delta)}{Z(t,T)}\right] &= 1 \\
E^t\left[\frac{P(t+\Delta)\cdot B(t)}{P(t,T)\cdot B(t+\Delta)}\right] &= 1 \\
E^t\left[\begin{array}{c} \exp\left\{-(\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[f(t+\Delta,j\Delta)-f(t,j\Delta)\right]\cdot\Delta)+f(t,t)\cdot\Delta\right\} \\ \cdot\exp\left\{-f(t,t)\cdot\Delta\right\} \end{array}\right] &= 1 \\
E^t\left[\exp\left\{-\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[f(t+\Delta,j\Delta)-f(t,j\Delta)\right]\cdot\Delta\right\}\right] &= 1 \qquad 0.59
\end{aligned}$$

Recall Equation (0.3), $f(t+\Delta,T)-f(t,T)=\alpha(t,T)\Delta+\sigma(t,T)X_1\sqrt{\Delta}$. Substituting Equation (0.59 d) with Equation (0.3), we have

$$E^t\left[\exp\left\{-\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[\alpha(t,j\Delta)\Delta^2+\sigma(t,j\Delta)X_1\Delta^{3/2}\right]\right\}\right]=1. \tag{0.60}$$

The risk-neutral drifts $\alpha$ are derived in terms of volatilities $\sigma$ :

$$\sum_{j=t/\Delta+1}^{T/\Delta-1}\alpha(t,j\Delta)=\frac{1}{\Delta^2}\ln(E^t\left[\exp\left\{-\sum_{j=t/\Delta+1}^{T/\Delta-1}\sigma(t,j\Delta)X_1\Delta^{3/2}\right\}\right]). \tag{0.61}$$

We come to the drifts $\beta(t,T)$. Recall Equation (0.56 a),

$$P^*(t,T)=\exp\left\{-\left[r(t)+s(t,t)\right]\cdot\Delta\right\}E^t\left[P^*(t+\Delta,T)\right]. \tag{0.62}$$

Rearranging the terms, we have

$$\begin{aligned} E^t\left[\frac{\exp\left\{-\varphi(t,t)\Delta\right\}P^*(t+\Delta,T)}{P^*(t,T)}\right] &= 1 \\ E^t\left[\begin{array}{c}\exp\left\{-(\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[\varphi(t+\Delta,j\Delta)-\varphi(t,j\Delta)\right]\cdot\Delta)+\varphi(t,t)\cdot\Delta\right\} \\ \cdot\exp\left\{-\varphi(t,t)\cdot\Delta\right\}\end{array}\right] &= 1 \\ E^t\left[\exp\left\{-\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[\varphi(t+\Delta,j\Delta)-\varphi(t,j\Delta)\right]\cdot\Delta\right\}\right] &= 1 \qquad 0.63 \end{aligned}$$

Recall Equation (0.3), (0.4) and definition of $\varphi$, we have

$$\begin{aligned} \varphi(t+\Delta,j\Delta)-\varphi(t,j\Delta) &= f(t+\Delta,j\Delta)+s(t+\Delta,j\Delta)-\left[f(t,j\Delta)+s(t,j\Delta)\right] \\ &= \alpha(t,T)\Delta+\beta(t,T)\Delta+\sigma(t,T)X_1\sqrt{\Delta}+\eta(t,T)X_2\sqrt{\Delta} \qquad 0.64 \end{aligned}$$

Substituting Equation (0.62 c) with Equation (0.63 b), we get

$$E^t\left[\exp\left\{-\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[\alpha(t,T)\Delta+\beta(t,T)\Delta+\sigma(t,T)X_1\sqrt{\Delta}+\eta(t,T)X_2\sqrt{\Delta}\right]\cdot\Delta\right\}\right]=1 \quad (0.65)$$

Rearranging gives $\beta$ in terms of $\alpha$, $\sigma$, and $\eta$ :

$$\sum_{j=t/\Delta+1}^{T/\Delta-1}\beta(t,j\Delta)=\frac{1}{\Delta^2}\ln\left\{E^t\left[\exp\left(-\Delta^{3/2}\sum_{j=t/\Delta+1}^{T/\Delta-1}\left[\sigma(t,j\Delta)X_1+\eta(t,j\Delta)X_2\right]\right)\right]\right\}-\sum_{j=t/\Delta+1}^{T/\Delta-1}\alpha(t,j\Delta). \quad (0.66)$$