Asset Prices and Real Exchange Rates with Deep Habits *

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Abstract

I study a two-country, two-good pure-exchange economy with deep habits and a consumption home bias. The model jointly accounts for the volatility of the real exchange rate, equity premiums, stock return volatilities, the level and the volatility of the risk free rates and reproduces the uncovered interest rate parity puzzle. Both the volatility of the real exchange rate and the equity premium depend on habit formation, but load differently on the preference parameters. While the equity premium mainly depends on risk aversion, the real exchange rate depends mostly on the elasticity of substitution between the home and the foreign good. I illustrate that a consumption home bias in traded goods leads to a portfolio home bias.

Keywords: Asset Pricing Moments; Real Exchange Rates; Multi-Good Economies; Deep Habits

JEL Classification: F31; G10
1 Introduction

To successfully account for asset prices in an international context, the stochastic discount factor in different countries must not only be able to price domestic assets, but also be consistent with exchange rate dynamics. Standard consumption based models fail to jointly match asset pricing moments such as the equity premium, the volatility of stock returns and the risk free rate and international quantities such as the volatility of the real exchange rate and the failure of the uncovered interest rate parity (UIP). I study the effect of deep habits on asset prices and on exchange rates in a Lucas (1982) two-country, two-good pure-exchange economy. I depart from the habit formation typically employed in the asset pricing literature in three ways: i) habit formation is formed at the individual country good level, ii) utility is non-separable over domestic and foreign goods and iii) agents have consumption home bias. In my model all goods are tradable, but agents prefer the domestically produced consumption good. Consumption home bias leads to real exchange rate movements driven by the terms of trade and portfolio home bias in traded goods. Combining habit formation at the country good level with non-separable utility functions allows for jointly matching domestic asset pricing moments such as equity premiums, stock return volatilities, the level and the volatility of the interest rates along with the volatility of the real exchange rate and the failure of UIP.

In a model with habit formation at the country good level agents are averse to scaling back on goods for which the habit level is close to consumption, and consequently the real exchange rate responds more to output shocks than in a model without deep habits. The elasticity of substitution between domestic and foreign goods governs the magnitude of such output shocks on the real exchange rate. Similarly as in a model with standard habit formation, deep habits at the country good level increases the market price of risk, and therefore helps matching the equity premium. The market price of risk depends on risk aversion and the real exchange rate volatility depends on the elasticity of substitution between home and foreign
goods. Non-separable utility functions allow for separating the elasticity of substitution between home and foreign goods from risk aversion, and thus help to match the volatility of the real exchange rate and the equity premium simultaneously.

UIP states that the expected change in the exchange rate equals the interest rate differential. Hence, a country with a high interest rate is expected to experience a depreciating exchange rate relative to a low interest rate country.\(^1\) However, empirical evidence shows that high interest rate countries experience appreciating rather than depreciating exchange rates (see Hansen and Hodrick (1980), Fama (1984), Backus et al. (2001)). To reproduce the failure of UIP under rational expectations, the risk premium must be negatively correlated with the interest rate differential and exhibit a higher volatility than the expected depreciation of the real exchange rate. My model can match the high volatility of the risk premium due to the time-varying degree of risk aversion induced by habit formation. In my model with deep habits, the market price of risk is counter-cyclical. In times when the surplus consumption ratio\(^2\) in the home good is lower than the foreign good, agents are reluctant to scale back on domestic good consumption, and therefore require a positive premium on the exchange rate. Moreover, if risk aversion is high compared to the elasticity of substitution between the home and the foreign good, then the interest rate differential is pro-cyclical. As the real exchange rate risk premium is counter-cyclical and the interest rate differential is pro-cyclical, the correlation between the two is negative thereby reproducing the UIP puzzle. I illustrate the failure of UIP in the model by simulating the real exchange rate and interest rates, and by replicating UIP regressions from the empirical literature. The slope and t-statistics are similar to the data, although the \(R^2\) is low.

Carry trades exploit the failure of UIP by taking long positions in high interest rate countries and short positions in low interest rate countries. To shed further light on the

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\(^1\) According to the UIP a regression of the exchange rate changes on the interest differentials should yield a slope coefficient of one.

\(^2\) The surplus consumption ratio is given by the consumption in excess of the habit level divided by consumption (see Campbell and Cochrane (1999)).
model’s ability to match the failure of UIP, I calculate the returns to carry trades in my model, and show that the model is able to generate excess returns. However, the Sharpe ratio generated by my model is lower than the Sharpe ratio in the data. This is mainly due to the low explanatory power of the interest rate differential in explaining real exchange rate changes.

"Catching up with the Joneses” or external habit formation captures the idea that utility is not only derived from current consumption, but from consumption relative to past aggregate consumption of some reference group. In times when the current consumption is low compared to the benchmark level of the reference group, then utility is low. External habit formation makes the utility function time non-separable. Empirical evidence has emphasized the importance of time non-separable preferences. In my model each agent benchmarks their consumption of each good against a function of past output of the same good. However, the home and the foreign agent put different weights on the benchmark of the two goods. Heterogeneous tastes and consumption benchmarks implies that I need to solve for the optimal allocations. Models with external habit formation primarily focus on representative agent economies, and few aggregation results exist. I show that under complete markets country allocations can be solved via a corresponding central planner problem. Furthermore, the optimal allocations of the habit-adjusted consumptions only depend on the aggregate habit-adjusted output of each good. This allows for modeling the aggregate surplus consumption ratios of the home and the foreign good directly as in standard representative agent economies. Although asset prices only depend on the aggregate habit-adjusted outputs, the optimal consumption and portfolio policies of the home and the foreign agent require the knowledge of the individual habit levels. Therefore, I assume that the habit levels of each good are perfectly correlated across agents. However, the weights the

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agents attach to aggregate habit levels differ. I also assume that the weights are linked to the degree of consumption home bias so that habit formation does not change the consumption allocations in the steady state.

Preference and demand shocks are commonly used in the international economics and finance literature.\(^5\) I illustrate that the model with deep habits can be expressed as a model with demand and preference shocks. While the demand shocks govern the marginal utility of aggregate consumption, the preference shocks change the optimal composition of domestic and foreign goods in the utility function. I assume that demand and preference shocks are perfectly correlated with output shocks. By assuming perfect correlation with output, I put more structure on the form of the preference and the demand shocks that are typically employed in the literature.

My model is related to several recent papers that study the joint behavior of exchange rates and asset prices. Colacito and Croce (2008) study a two-country version of the long-run risk model of Bansal and Yaron (2004). When the long-run component in consumption growth is perfectly correlated across the home country and the foreign country, then the volatility of the real exchange rate and equity premiums are matched simultaneously. Their model, however, cannot match the failure of UIP as it generates a constant market price of risk. Bansal and Shaliastovich (2008) also study a two-country long-run risk model. When the home country and foreign country consumption volatility is stochastic, then the model reproduces the failure of the expectation hypothesis in both the bond market and the exchange rate market. Verdelhan (2008) uses the model of Campbell and Cochrane (1999) to study the failure of UIP. The model replicates the equity premium, the risk free rate, and deviations from UIP at the expense of a too volatile real exchange rate.\(^6\) Common to the models by Colacito and Croce (2008), Bansal and Shaliastovich (2008), and Verdelhan

\(^5\)Dornbusch et al. (1977), Pavlova and Rigobon (2007) and Pavlova and Rigobon (2008) consider models with preference shocks or stochastic demand shifts in an international setting.

\(^6\)In the base case model of Verdelhan (2008) the volatility of the real exchange rate is 42% but the data shows only an average exchange rate volatility of about 12%. Verdelhan (2008) shows that including trade costs can reduce the volatility of the real exchange rate.
(2008) is that they all specify a separate pure-exchange economy for the home country and the foreign country. Consumption in the two countries is exogenously specified. In a closed pure-exchange economy, such consumption allocations require that the consumption goods are non-tradable or that there is complete home bias. In my model the countries produce different goods that are less than perfect substitutes. Home and foreign goods are tradable, so agents trade to share consumption risk. Consumption allocations differ across agents because agents attach greater weights to domestically produced consumption goods. Making the consumption allocation endogenous introduces interdependences between the stochastic discount factor in the home and the foreign country. This leads to excess correlations, volatility spillovers, and correlated risk free rates. As opposed to Verdelhan (2008) and Bansal and Shaliastovich (2008), the real short rate does not have to be pro-cyclical in order to match the failure of UIP.

My model is also related to Moore and Roche (2008). They consider a Lucas (1982) economy with separable power utility for the foreign and the home consumption good. They only examine the properties of the exchange rate, and do not take into account how the model fits other asset pricing moments. Their model matches several features of the real exchange rate, but cannot jointly match the equity premium and the exchange rate volatility due to the assumption of separable power utility. The models of Moore and Roche (2008) and Verdelhan (2008) yield the same properties for the real exchange rate and asset pricing moments. In Verdelhan (2008), the assumption is that the representative agent in each country only cares about domestic consumption resulting in complete home bias with standard habit formation. In contrast, the model of Moore and Roche (2008) assumes that all goods are traded and that preferences are homogeneous.\footnote{My model nests Moore and Roche (2008) and Verdelhan (2008) on the preference side. If the elasticity of substitution between the home good and the foreign good is the reciprocal of the risk aversion or there is complete home bias then my model collapses to a separable power utility over the two goods. The economic interpretation of my model is closer to Moore and Roche (2008) as they also consider a multiple good setting with deep habits. However, while I allow for preference heterogeneity, Moore and Roche (2008) assume homogeneous preferences.}
Pavlova and Rigobon (2007) study a model with log-linear preferences and demand shocks.\textsuperscript{8} They show that demand shocks can be important for the covariance of stock, bond and exchange rate markets. My model extends their model on the preference side by allowing for more general utility functions. However, their specifications of demand shocks are more general in that they allow for pure demand shocks uncorrelated with other fundamentals.

Kollmann (2006) studies the implication of home bias in consumption on portfolio policies. He assumes that all goods are tradable, but agents prefer the domestically produced consumption good. If the elasticity of substitution between home and foreign goods does not exceed one, then the value of the home output relative to the foreign output falls after a positive shock to the home output due to the decline in the relative price. Moreover, when there is consumption home bias it is optimal to reduce the locally consumed fraction of the home good after a positive shock. The home country stock market works as a hedge against the local consumption fraction of domestic output, and consequently the model generates portfolio home bias. I show that a similar mechanism is at work in my model with deep habits. After a shock to the habit adjusted consumption of the home good, the home agent optimally reduces the consumption fraction of the local good. The response of the relative price of the home and the foreign good makes it optimal to hold a fraction of the domestic stock market that exceeds the locally consumed fraction of domestic output.

Ravn et al. (2006) study the implications of good specific habit formation on markups. They label this type of habit formation deep habits as opposed to superficial habits. In an asset pricing context van Binsbergen (2009) uses the model of Ravn et al. (2006) to study the cross-sectional properties of stock returns. I depart from the previous deep habits literature by assuming that agents are heterogeneous. I do so by assuming a pure-exchange economy instead of a production economy employed by Ravn et al. (2006) and van Binsbergen (2009).

The rest of the paper is organized as follows. Section 2 describes the model and its equilibrium. Section 3 numerically examines equilibrium properties. Section 4 concludes. Appendix A derives the equilibrium. Appendix B and C deal with the Malliavin derivatives. Appendix D presents the numerical method used to solve for equilibrium. Finally, Appendix E discusses how the choice of numeraire impacts the equilibrium.

2 The Model

In this section I specify the economy and then derive the equilibrium. All proofs are relegated to Appendix A. My model is an extension of the Lucas (1982) two-country model to include deep habits. I focus on real quantities and therefore do not include nominal quantities.\(^9\)

2.1 The Economy

I consider a continuous time pure exchange economy over the time span \([0, T]\). The uncertainty is represented by a filtered probability space \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})\), on which is defined a two-dimensional Brownian motion \(B = (B_1, B_2)\). In the following all stochastic processes are assumed to be progressively measurable and all equalities are assumed to hold in the almost surely sense. Stochastic differential equations are assumed to have solutions without stating the regularity conditions.

There are two countries in the world economy. Each country produces its own perishable consumption good. Output of each good follows

\[
\frac{d\delta_i(t)}{\delta_i(t)} = \mu_{\delta_i} dt + \sigma_{\delta_i}^\top dB(t)
\]

for \(i = H, F\) where \(\top\) denotes the transpose, \(H\) denotes the home country, and \(F\) denotes the foreign country. The diffusion coefficients are two-dimensional vectors. In this way, output

\(^9\)As my goal is to study the real exchange rate I do not employ cash in advance as in Lucas (1982). The real exchange rate is unaffected by this simplification.
across the two countries is allowed to be correlated. Investment opportunities consists of a bond in zero net supply paying out in the home habit-adjusted composite good, a bond in zero net supply paying out in the foreign habit-adjusted composite good, and stock markets in both countries. Stocks are in unit supply and represent claims to each country’s respective output stream. I use the habit-adjusted composite good of the home country as numeraire. The state price density, $\xi(t) = \xi_H(t)$, is defined in terms of the home country’s habit-adjusted composite basket. The equilibrium state price density process follows

$$d\xi(t) = \xi(t) \left( -r_H(t) dt - \theta_H(t)^\top dB(t) \right)$$

(2)

with $\xi(0) = 1$, where $\theta_H(t)$ denotes the market price of risk. We can similarly define the foreign country state price density

$$d\xi_F(t) = \xi_F(t) \left( -r_F(t) dt - \theta_F(t)^\top dB(t) \right).$$

(3)

Under complete markets the ratio of the foreign to the home country state price density uniquely defines the real exchange rate.

The bond price dynamics are given by

$$dB_i(t) = r_i(t) B_i(t) dt$$

(4)

In Equation (4) the bond price dynamics in country $i$ is measured in terms of the composite good of the same country.

Prices of the home good and the foreign good measured in terms of the home country

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10One of the two bonds is redundant since we have two sources of uncertainty and four assets. However, I calculate both the home and the foreign risk free rates to study UIP.
habit-adjusted composite good follow

\[ dP_i(t) = P_i(t) \left( \mu_{P_i}(t) dt + \sigma_{P_i}(t)^\top dB(t) \right). \] (5)

The real exchange rate follows

\[ de(t) = e(t) \left( \mu_e(t) dt + \sigma_e(t)^\top dB(t) \right). \] (6)

Stock price dynamics are given by

\[ dS_i(t) + P_i(t) \delta_i(t) dt = S_i(t) \left( \mu_i(t) dt + \sigma_i(t)^\top dB(t) \right) \] (7)

Coefficients for bond prices, stock prices, commodity prices and the real exchange rate are determined in equilibrium. The market price of risk is related to the bond and stock price dynamics by

\[ \theta_H(t) = \sigma(t)^{-1} (\mu(t) - r_H(t) I) \] (8)

where \( \sigma \) represents a \( 4 \times 4 \) matrix containing the stock price diffusion coefficients, \( \mu \) is a vector of drift rates and \( I \) denotes a vector of ones.

Each country is populated by a representative agent that maximizes lifetime expected utility from consumption of the two goods

\[
\max_{C_H^j, C_F^j, \pi^j, \varphi^j} \mathbb{E} \left[ \int_0^T u_j \left( C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t), t \right) dt \right] \] (9)

where

\[ u_j \left( C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t), t \right) = \frac{e^{-\rho t}}{1-\gamma} Z_j \left( C_H^j(t), C_F^j(t), X_H^j(t), X_F^j(t) \right)^1 - \gamma \] (10)
and

\[ Z_j(C^j_H(t), C^j_F(t), X^j_H(t), X^j_F(t)) = \left( \phi^j_H - \beta \right)(C^j_H(t) - X^j_H(t))^{(1 - \beta)} + \left( 1 - \phi^j_H \right)(1 - \beta) \left( C^j_F(t) - X^j_F(t) \right)^{(1 - \beta)} \right) \]  

subject to the dynamic budget constraint

\[ dW^j(t) = \varphi^j_H(t) \frac{dH(t)}{H(t)} + \pi^j_H(t) \frac{dS_H(t) + P_H(t)\delta_H(t)}{S_H(t)} + \frac{\pi^j_F(t) dS_F(t) + P_F(t)\delta_F(t)}{S_F(t)} - P(t)^{T} C^j(t) dt \]  

for \( j = H, F \), where \( \pi^j(t) = (\pi^j_H(t), \pi^j_F(t)) \) represents the vector of amounts held in the stocks by agent \( j \). It is assumed that \( W^j(0) > 0 \) with \( W^j(0) = \pi^j(0)^{T} I \), i.e., the agents are endowed with shares in the stocks. The vector of the country specific good prices is \( P(t) = (P_H(t), P_F(t))^{T} \) and \( \varphi^j_H \) denotes the amount invested by agent \( j \) in the bond of the home country.\(^{11}\) I impose the symmetry condition \( \phi_F = 1 - \phi_H \). To simplify notation I occasionally drop the subscript and denote \( \phi = \phi_H \).

The above utility function is motivated by Ravn et al. (2006), where \( C^j_i \) is the optimal consumption of the agent from country \( j \) of goods from country \( i \), and \( X^j_i \) is the habit level of the same good. Instead of forming habits over an aggregate consumption bundle, the agents form habits over each country good variety. The agents do not take into account the habit level in their optimization and thus habit is external. The deep habit formulation can be interpreted as a special case of demand and preference shocks. Rewrite the composite good of agent \( j \) as

\[ Z_j(C^j_H(t), C^j_F(t), \eta^j_1(t), \eta^j_2(t)) = \eta^j_1(t) \left( \phi^j_H - \beta \right)(C^j_H(t))^{(1 - \beta)} + \left( 1 - \phi^j_H \right)(1 - \beta) \eta^j_2(t) \left( C^j_F(t) \right)^{(1 - \beta)} \right) \]  

where \( \eta^j_1(t) = \frac{C^j_H(t) - X^j_H(t)}{C^j_H(t)} \) and \( \eta^j_2(t) = \frac{C^j_F(t) - X^j_F(t)}{C^j_F(t)} \). In Equation (13), \( \eta^j_2 \) is a demand

\(^{11}\)As one of the two bonds are redundant, I only include the bond denominated in the numeraire good.
shock governing utility from aggregate consumption, and \( \eta^j_2 \) is a preference shock that shifts the relative importance of the two goods. Specifying the utility function as in Equation (11) impose structure on the nature of the demand shocks, as the shocks to the habit levels are perfectly correlated with output shocks. Utility is defined as a standard power utility function over the composite good \( Z \),\(^{12}\) with a time discount factor of \( \rho \). The composite good \( Z \) captures the degree of substitutability between the two goods in the economy. The functional form is a constant elasticity of substitution (CES) aggregator over the habit-adjusted consumption. When \( \beta = 1 \) the goods are perfect substitutes. For \( \beta = 0 \) the Cobb-Douglas utility function over the habit-adjusted consumption of the two goods is obtained.\(^{13}\) The parameter \( \phi_i \) is the weight that each agent attach to the home good and is the only source of preference heterogeneity. There is consumption home bias if \( \phi > \frac{1}{2} \).

### 2.2 Solving the Model

To derive equilibrium, I use standard martingale techniques (see Cox and Huang (1989), Karatzas et al. (1990)).

Next I define equilibrium.

**Definition 1.** Equilibrium is a collection of allocations \( (C^j_H, C^j_F, \varphi^j_H, \pi^j_H, \pi^j_F) \) for \( j = H, F \), and a price system \( (\mu, \mu_P, \sigma, \sigma_P, r_H) \), such that \( (C^j_H, C^j_F, \varphi^j_H, \pi^j_H, \pi^j_F) \) are optimal solutions to agent \( j \)'s optimization problem and good and financial markets clear

\[
C^H_H(t) + C^F_H(t) = \delta_H(t) \tag{14}
\]

\[
C^H_F(t) + C^F_F(t) = \delta_F(t) \tag{15}
\]

\[
\pi^H_H(t) + \pi^F_H(t) = S_H(t) \tag{16}
\]

\(^{12}\)Risk aversion in a multiple good setting with deep habits is non-trivial, however I will frequently refer to \( \gamma \) as the risk aversion.

\(^{13}\)The elasticity of substitution between the home good and the foreign good, \( \eta \), is related to \( \beta \) by \( \eta = \frac{1}{1-\beta} \).
\[ \pi_H^F(t) + \pi_F^F(t) = S_F(t) \]  
(17)

\[ \varphi_H^F(t) + \varphi_F^H(t) = 0 \]  
(18)

for \( t \in [0, T] \).

### 2.2.1 Optimal Consumption Allocations

Under complete markets\(^\text{14}\) we can solve the corresponding central planner problem at each time and state

\[
U(\delta(t), X(t), t) = \max_{C_H(t), C_F(t)} \left\{ \begin{array}{c}
a \ u_H(C_H(t), X^F(t), t) + \\
(1 - a)u_F(C_F(t), X^F(t), t)
\end{array} \right\}
\]  
(19)

subject to \( C_H(t) + C_F(t) = \delta(t) \). In Equation (19), let \( \delta(t) = (\delta_H(t), \delta_F(t)) \), \( C^i(t) = (C^i_H(t), C^i_F(t)) \), \( X(t) = (X_H(t), X_F(t)) \), \( X^i(t) = (X^i_H(t), X^i_F(t)) \) and the weight \( a \) represents the home agent’s weight in the central planner problem with a one-to-one mapping with initial endowments in the stocks. The aggregate habit level of good \( i \) in the central planner problem is defined as

\[ X_i(t) = X_i^H(t) + X_i^F(t). \]  
(20)

Write the resource constraint as

\[ C_i^H(t) - X_i^H(t) + C_i^F(t) - X_i^F(t) = \delta_i(t) - X_i(t). \]  
(21)

\(^\text{14}\)There are four securities and two Brownian motions, consequently the market is potentially complete. However, as market completeness depends on the endogenous stock price diffusion coefficients the market completeness must be verified after calculating the stock price diffusion matrix.
By the implicit function theorem it can be shown that the habit-adjusted consumption of good \( i \) of agent \( j \) is given by

\[
Q^j_i(t) = C^j_i(t) - X^j_i(t) = g^j_i(Q_H(t), Q_F(t))
\]  

(22)

where \( g^j_i : R_{++}^2 \to R_{++} \) and \( Q_j = \delta_j(t) - X_j(t) \) for \( i, j = H, F \). Note that the utility function of the representative agent only depends on aggregate habit adjusted consumption of each good. As long as the aggregate habit of good \( i \) is less than aggregate output of the same good, the habit-adjusted consumption of each agent is guaranteed to stay positive, and the optimization problem is well defined.

2.2.2 The Habit Processes

The central planner problem only determines the habit-adjusted consumption of each agent. To derive the total consumption one must make assumptions about the individual habit levels \( X^j_i \). I assume that

\[
X^H_H(t) = \phi X^H_H(t)
\]  

(23)

\[
X^F_H(t) = (1 - \phi) X^H_H(t)
\]  

(24)

\[
X^H_F(t) = (1 - \phi) X^F_F(t)
\]  

(25)

\[
X^F_F(t) = \phi X^F_F(t).
\]  

(26)

This implies that the habit level of each good is perfectly correlated across the agents. Agents use the same habit process, but attach different weights to it. The weight they attach to the habit process is determined by \( \phi \), appearing in the composite good. If there is home bias in the utility function, then the home agent attaches greater weight to the habit process for the home good than the foreign agent. Moreover, when the two countries are symmetric, then the consumption allocations are the same in the steady state as in the case without habit
formation. Thus, linking the weights in the habit processes to the weight in the composite good does not introduce an additional home bias in the symmetric case.\textsuperscript{15}

As the habit levels of each agent are perfectly correlated, one can directly model the aggregate habit levels $X_i$. As in Campbell and Cochrane (1999), define the surplus consumption ratio for good $i = H, F$ as

$$s_i(t) = \frac{\delta_i(t) - X_i(t)}{\delta_i(t)}$$

and assume that $s_i$ follows

$$ds_i(t) = \alpha_i (\overline{s}_i - s_i(t)) dt + s_i(t)\lambda_i (s_i(t)) \sigma_\delta^T dB(t)$$

where

$$\lambda_i (s_i(t)) = \sqrt{\frac{1 - s_i}{\overline{s}_i}} \sqrt{\frac{1 - s_i(t)}{s_i(t)}}.$$\textsuperscript{16}

The variable $s_i$ is a mean reverting process with long-run mean $\overline{s}_i$ and speed of mean reversion of $\alpha_i$. The process is locally perfectly correlated with output shocks to good $i = H, F$. To understand the dynamics of the surplus consumption ratio, consider the case when $X_i(t)$ is an exponential weighted average of past consumption\textsuperscript{16} of good $i$

$$X_i(t) = X_i(0)e^{-\kappa t} + \kappa \int_0^t e^{-\kappa(t-u)}C_i(u)du.$$\textsuperscript{17}

An application of Ito’s lemma on $s_i(t) = \frac{\delta_i(t) - X_i(t)}{\delta_i(t)}$ yields\textsuperscript{17}

$$ds_i(t) = (\mu_{\delta_i}(t) - \kappa - \sigma_\delta^T \sigma_\delta)(\frac{\mu_{\delta_i}(t) - \sigma_\delta^T \sigma_\delta}{\mu_{\delta_i}(t) - \kappa - \sigma_\delta^T \sigma_\delta} - s_i(t)) + (1 - s_i(t)) \sigma_\delta^T dB(t).$$\textsuperscript{17}

\textsuperscript{15}To see this note that when $Q_H = Q_F$ and $\delta_H = \delta_F$ then the optimal allocation of the habit-adjusted consumption is $Q_{H}^H = \phi_H Q_H$ and $Q_{F}^H = (1 - \phi_H)Q_F$. The consumption allocations are $C_{H}^H = \phi_H Q_H + \phi_H X_H = \phi_H \delta_H$ and is $C_{F}^H = (1 - \phi_H)Q_H + (1 - \phi_H)X_H = (1 - \phi_H)\delta_F$, resulting in the same allocations as in a model without habit formation where $\delta_H = \delta_F$.

\textsuperscript{16}Constantinides (1990) as well as Detemple and Zapatero (1991) model habits as an exponential weighted average of past consumption.

\textsuperscript{17}See Santos and Veronesi (2009) for a similar illustration.
The surplus consumption ratio is a mean reverting process, locally perfectly correlated with output shocks to good $i$. However, nothing prevents the process from becoming negative. To bound the surplus consumptions away from zero, I use the representation in Equation (28). The functional form of the sensitivity function, $\lambda_i$, differs from Campbell and Cochrane (1999) and follows Aydemir (2008). Compared to Campbell and Cochrane (1999) the surplus consumption ratio in Equation (28) put less weight on the boundaries of the distribution, thus fewer Monte Carlo simulations are necessary. The process is guaranteed to stay within the boundaries of $[0, 1]$ for a large set of parameters values.  

An application of Ito’s lemma gives the process followed by the habit-adjusted consumption $Q_i$

$$dQ_i(t) = Q_i(t) \left( \mu_{Q_i}(t)(t)dt + \sigma_{Q_i}(t)^\top dB(t) \right)$$  \hspace{1cm} (32)$$

where

$$\mu_{Q_i}(t) = \mu_{\delta_i} + \phi_i \left( \frac{\overline{s_i}}{s_i(t)} - 1 \right) + \lambda_i (s_i(t)) \sigma^\top_{\delta_i} \sigma_{\delta_i}$$  \hspace{1cm} (33)$$

and

$$\sigma_{Q_i}(t) = (1 + \lambda_i (s_i(t))) \sigma_{\delta_i}.$$  \hspace{1cm} (34)$$

In times when the surplus consumption ratio is low, then the sensitivity function $\lambda_i (s_i(t))$ is high, and so is the volatility of the habit-adjusted consumption. When the habit level is close to aggregate dividends, then the drift is high as the surplus consumption ratio is expected to revert back to its long-run mean. Given the process for the habit-adjusted dividends and the optimal allocations in Equation (22), I can solve for the process followed by the individual habit-adjusted consumptions and the composite goods. By Ito’s lemma

$$dZ_i(t) = Z_i(t) \left( \mu_{Z_i}(t)dt + \sigma_{Z_i}(t)^\top dB(t) \right)$$  \hspace{1cm} (35)$$

\hspace{1cm} \text{\\textsuperscript{18}}

\hspace{1cm} To guarantee that $s$ stays within the boundaries one requires that $a > 1$ and $b > 1$ with $a = \frac{2\epsilon_i \sigma_i^2}{\sigma_{\delta_i}^2 \sigma_{\delta_i} (1 - \bar{s}_i)}$ and $b = \frac{2\epsilon_i \sigma_i}{\sigma_{\delta_i}^2 \sigma_{\delta_i}}$ (see Aydemir (2008) for further details).
where

\[
\mu_{Z_i}(t) = s_i^\phi(t)\mu_{Q_i^H}(t) + \left(1 - s_i^\phi(t)\right)\mu_{Q_i^F}(t) \\
- \frac{1}{2} (1 - \beta) s_i^\phi(t) \left(1 - s_i^\phi(t)\right) \sigma_{Q_i^H}(t)\sigma_{Q_i^H}(t)^\top \\
- \frac{1}{2} (1 - \beta) s_i^\phi(t) \left(1 - s_i^\phi(t)\right) \sigma_{Q_i^F}(t)\sigma_{Q_i^F}(t)^\top \\
+ (1 - \beta) s_i^\phi(t) \left(1 - s_i^\phi(t)\right) \sigma_{Q_i^H}(t)\sigma_{Q_i^F}(t)^\top \sigma_{Q_i^F}(t) \tag{36}
\]

and

\[
\sigma_{Z_i}(t) = s_i^\phi(t)\sigma_{Q_i^H}(t) + \left(1 - s_i^\phi(t)\right)\sigma_{Q_i^F}(t). \tag{37}
\]

In the above

\[
s_i^\phi(t) = \frac{\phi_i^{1-\beta}Q_i^H(t)^\beta}{\phi_i^{1-\beta}Q_i^H(t)^\beta + (1 - \phi_i)^{1-\beta}Q_i^F(t)^\beta} \tag{38}
\]

where the expressions for \(\mu_{Q_i^j}\) and \(\sigma_{Q_i^j}\) can be found in Appendix A. The diffusion of the habit-adjusted consumption is a weighted average of the habit-adjusted consumption of the home and the foreign good. The weight, \(s_i^\phi\), is monotonically increasing in the degree of the consumption home bias parameter \(\phi\). In the limit \(\phi = 1\), i.e., with complete consumption home bias, the weight is one. In this case the correlation between the habit-adjusted consumption of the two agents are given by the output correlation between the home and the foreign good. Similarly, the drift of the habit-adjusted consumption collapses to the drift of the habit-adjusted output when there is complete home bias.
2.2.3 Equilibrium

Define the consumption-based price index consistent with a constant elasticity of substitution utility function as (see Obstfeld and Rogoff (1996))

\[
P^H(t) = \left( \phi P^H(t) \frac{\beta}{1-\beta} + (1 - \phi) P_F(t) \frac{\beta}{1-\beta} \right)^{\frac{1-\beta}{\beta}}
\]

\[
P^F(t) = \left( (1 - \phi) P^H(t) \frac{\beta}{1-\beta} + \phi P_F(t) \frac{\beta}{1-\beta} \right)^{\frac{1-\beta}{\beta}}.
\]

(39)

The real exchange rate is defined as the ratio of the foreign country price index to the home country price index

\[
e(t) = \left( \frac{1 - \phi}{\phi} + \phi \tau(t) \frac{\beta}{1-\beta} \right)^{\frac{1-\beta}{\beta}}
\]

(40)

where \(\tau(t) = \frac{P_F(t)}{P^H(t)}\) is the relative price of the foreign good in terms of the home good. Note that an improvement of the home country terms of trade corresponds to a decrease in \(\tau\).

The equilibrium relative price of the foreign good in terms of the home good is

\[
\tau(t) = \frac{\partial u_i(C^H(t), C^F(t), X^H(t), X^F(t), t)}{\partial C^F(t)} \frac{\partial u_i(C^H(t), C^F(t), X^H(t), X^F(t), t)}{\partial C^H(t)}
\]

\[
= \left( \frac{1 - \phi}{\phi} \right)^{1-\beta} \left( \frac{Q^H(t)}{Q^F(t)} \right)^{1-\beta}.
\]

(41)

Since I am using the habit-adjusted consumption index as numeraire, one has from the first order conditions that the equilibrium state price density in the home country is

\[
\frac{\xi(t)}{\xi(0)} = \frac{\xi_H(t)}{\xi_H(0)} = e^{-\rho t} \left( \frac{Z_H(t)}{Z_H(0)} \right)^{-\gamma}.
\]

(42)

\(^{19}\)I use subscripts to denote good specific prices and superscript to denote price indexes, i.e., \(P^H\) denotes the price of the good produced by country H and \(P^H\) denotes the consumption price index of the same country.
Similarly, the foreign state price density is

\[
\frac{\xi_F(t)}{\xi_F(0)} = e^{-\rho t} \left( \frac{Z_F(t)}{Z_F(0)} \right)^{-\gamma}.
\] (43)

The next proposition characterizes the equilibrium real short rate and the market price of risk in the home and the foreign country.

**Proposition 1.** If equilibrium exists, then the real short rate in the home and the foreign country are

\[
r_i(t) = \rho + \gamma \mu Z_i(t) - \frac{1}{2} \gamma (\gamma + 1) \sigma Z_i(t)^\top \sigma Z_i(t).
\] (44)

The market prices of risk are

\[
\theta_i(t) = \gamma \sigma Z_i(t)
\] (45)

for \(i = H, F\).

Note that the short rates and the market prices of risk take the usual form as in the standard power utility case with the exception that consumption is replaced with the aggregate habit-adjusted consumption \(Z\). The market prices of risk are time-varying as the volatility of habit-adjusted consumptions depend on the surplus consumption ratios.

**Remark 1.** In the case of complete consumption home bias, i.e., \(\phi = 1\), the risk free rates and the market prices of risk are

\[
r_i(t) = \rho + \gamma \mu Q_i(t) - \frac{1}{2} \gamma (\gamma + 1) \sigma Q_i(t)^\top \sigma Q_i(t)
\]

\[
\theta_i(t) = \gamma \sigma Q_i(t).
\] (46)

In the special case presented in Remark 1 combined with i.i.d. output the short rate in the home and the foreign country will be uncorrelated. Moreover, the market price of risk in the home (foreign) country only depends on the home (foreign) surplus consumption ratio.
In the more general case, the market prices of risk and the short rates will depend on both surplus consumption ratios.

The next proposition describes the real exchange rate process.

**Proposition 2.** If equilibrium exists, then the drift and the diffusion coefficients of the real exchange rate process in Equation (6) are

\[ \mu_e(t) = r_H(t) - r_F(t) + \theta_H(t) \dot{\sigma}_e(t) \]  

(47)

and

\[ \sigma_e(t) = \theta_H(t) - \theta_F(t). \]  

(48)

The diffusion coefficients of the real exchange rate in Equation (48) are given by the difference in the market prices of risk in the home and the foreign country. As the market prices of risks are time-varying, the volatility of the real exchange rate will be stochastic. From Equation (47) we see that the expected change in the real exchange rate can be decomposed into the interest rate differential, \( r_H(t) - r_F(t) \) and the risk premium \( \theta_H(t) \dot{\sigma}_e(t) \).

The risk premium on the real exchange rate is given by the covariance of the real exchange rate with the domestic pricing kernel. Exchange rate volatility that is uncorrelated with the domestic pricing kernel does not command a risk premium. As long as the volatility of the real exchange rate is different from zero, then the foreign exchange rate risk must be priced from both the domestic and the foreign point of view.\(^{20}\)

**Remark 2.** In the case of complete consumption home bias, i.e., \( \phi = 1 \), and i.i.d. output growth, the real exchange rate risk premium from the home agents point of view is

\[ \theta_H(t) \dot{\sigma}_e(t) = (1 + \lambda(s_H(t)))^2 \sigma_H. \]  

(49)

From Remark 2 one can see that in the special case of complete home bias and i.i.d.\(^{20}\)}
output growth, the risk premium is only a function of the home country surplus consumption ratio. In times when the surplus consumption ratio is low, the sensitivity function $\lambda(s_H(t))$ is high and consequently the exchange rate risk premium is high. In the case of complete consumption home bias and i.i.d. output growth, then a necessary condition for a negative UIP slope coefficient is pro-cyclical real interest rates. This follows from the fact that the exchange rate premium is counter-cyclical, and to match the failure of the UIP the interest rate differential and the exchange rate premium must be negatively correlated. In the general case, this is not necessary as both the real short rates and the exchange rate premium are functions of the surplus consumption ratios in both countries.

The next proposition characterizes the equilibrium stock price diffusion matrix and expected returns. I follow Gallmeyer (2002) and apply the Clark-Ocone theorem from Malliavin Calculus to obtain explicit formulas for the stock price diffusion coefficients.

**Proposition 3.** If equilibrium exists, then the stock price diffusion coefficients are given by

$$\sigma_{S_i}(t) = \theta_H(t) + \frac{E_t \left[ \int_t^T \xi(u) P_i(u) \delta_i(u) \Phi_i(t,u) du \right]}{E_t \left[ \int_t^T \xi(u) P_i(u) \delta_i(u) du \right]}$$  \hspace{1cm} (50)

where

$$\Phi_i(t,u) = D_t \ln \xi(u) + D_t \ln P_i(u) + D_t \ln \delta_i(u).$$  \hspace{1cm} (51)

Moreover, the expected returns are

$$\mu_i(t) = r_H(t) + \theta_H(t)^\top \sigma_{S_i}(t)$$  \hspace{1cm} (52)

for $i = H, F$.

In Proposition 3, $D_t$ denotes the Malliavin derivative. The stock price diffusion matrix for country $i$ depends on the market price of risk and a ratio of expectations involving Malliavin derivatives. Consider the Malliavin derivative in the integrand
\[ \xi(u)P_i(u)\delta_i(u) (D_t \ln \xi(u) + D_t \ln P_i(u) + D_t \ln \delta_i(u)). \]  

This Malliavin derivative captures the response to a small change at time \( t \) to the state price density, the price and output of good \( i \) at a future time \( s > t \). The explicit calculation of the Malliavin derivatives can be found in Appendix B.

**Remark 3.** In the case of complete consumption home bias, i.e., \( \phi = 1 \), and i.i.d. output growth, then the correlation between stock returns measured in the local currency is zero.

Remark 3 illustrates that in the special case of complete home bias, there is no spillover correlation between returns in local currencies. This is a consequence of zero correlation between the stochastic discount factor in the home and the foreign country. In the general case, the stochastic discount factors are correlated due to consumption sharing, and consequently stock return in local currencies will be correlated.

The next proposition characterize the optimal portfolio of agent \( j = H,F \).

**Proposition 4.** If equilibrium exists, then the optimal portfolio policies are given by

\[ \pi^i(t) = GOP^i(t) + \phi_i \Delta_H(t) + (1 - \phi_i) \Delta_F(t) + \Psi^i(t) \]  \hspace{1cm} (54)

where

\[ GOP^i(t) = (\sigma_S(t)^T)^{-1} \theta_H(t) W^i(t) \]

\[ \Delta_j(t) = (\sigma_S(t)^T)^{-1} \xi(t)^{-1} \mathbb{E}t \left[ \int_t^T D_t (\xi(u)P_j(u)X_j(u)) \, du \right] \]

\[ \Psi^i(t) = (\sigma_S(t)^T)^{-1} \xi(t)^{-1} \mathbb{E}t \left[ \int_t^T D_t (\xi(u) (P_H(u)Q_H^i(u) + P_F(u)Q_F^i(u))) \, du \right] \]

for \( i,j = H,F \).

From Proposition 4 one can see that the portfolio policies can be decomposed into four parts. The first term, \( GOP^i \), is the growth optimal portfolio. This corresponds to the
optimal portfolio of an agent with logarithmic utility function in a single good economy. The second and the third term are adjustments for the habit levels. Note that the only difference between the two agents with respect to the second and the third terms are the loadings. The home agent loads more onto the term corresponding to the home good habit level when there is consumption home bias. The last term is correcting for the habit-adjusted consumptions.

**Remark 4.** In the case of complete consumption home bias, i.e., $\phi = 1$, then there is complete portfolio home bias. Moreover, in the case of no consumption home bias, i.e., $\phi = 0.5$ and equal initial endowments, then the portfolio policies are the same for the home and the foreign agents.

Remark 4 shows that the consumption home bias and portfolio home bias are the same in the extreme cases.

## 3 Numerical Analysis and Results

In this section I numerically study properties of the equilibrium when calibrated to the data.

### 3.1 Data and Calibration

I calibrate the model to US and UK data. The output processes are calibrated to real GDP data. The nominal GDP and CPI data are taken from OECD and IMF respectively, and consist of quarterly observations from 1971 to 2008. GDP is deflated using the CPI. Table 1 shows the summary statistics of the annualized real GDP and CPI data. Nominal short rates are approximated by 3 month government bills. To calculate real short rates I adjust the nominal short rates by expected inflation, where the expected inflation is estimated by a Kalman filter assuming that expected inflation is an AR(1) process in both countries. Table 2 show the estimated parameters based on a MLE estimation of the Kalman filters and Figure 1 shows the estimated expected inflations together with realized inflations. Nominal
exchange rates are from WM/REUTERS. To calculate real exchange rates I deflate the nominal exchange rates by the realized inflation in each country as measured by the CPI growth. For the stock market data I use the total return indexes from Datastream.

Table 3 summarizes the model parameters. For the risk aversion I use a coefficient of four. This is higher than in the standard external habit literature where the typical value is two, but lower than the value used by van Binsbergen (2009). I set the steady state value of the habit level to 0.15 and the persistence of the habit level to 0.04 in both countries. Time discount factor is 0.07 and chosen to match the level of the risk free rate. For the parameters of the dividend processes I calibrate my model to the average of the US and UK GDP data. The output processes have an expected growth rate of 2.4% and a standard deviation of 2.3%. The degree of consumption home bias, \( \phi \), is calibrated so that \( \text{Import}_i/GDP_i = 0.15 \) for \( i = H, F \). This is motivated by Backus et al. (1994). For all economies the time horizon is 100 years.

Table 4 summarizes the key moments in the baseline calibration. The model excess returns are 4.8% in the US and 5.7% in the UK.\(^{21}\) The corresponding values in the data are 6.6% and 8.7% respectively. Note that the model implied standard deviation of the market returns are somewhat low compared to the data. In the model the standard deviations are 9.9% and 16.9% compared to 15.5% and 19.8% in data for the US and UK respectively. The model implied risk free rate is 1.3% compared to 1.7% for the US and 2.6% for the UK data. As one can see, the model is able to resolve the risk free rate puzzle. The standard deviation of the risk free rate is 3.0% in the model compared to 2.1% for US and 3.2% for UK. The model implied correlation between US and UK returns is close to what one see in the data with a value of 0.572 compared to 0.61. The standard deviation of the real exchange rate is 0.136 compared to 0.11 in the data. The autocorrelation of the model implied exchange rate is the same as in the data. The fraction of the wealth invested in the domestic market

\(^{21}\) Returns are measured in US dollar.
is 0.885. This is higher than the corresponding consumption home bias which is 0.85. For US investors the fraction of wealth invested in the domestic market is 0.883.\textsuperscript{22}

### 3.2 Real Exchange Rate Volatility

Figure 2 shows the volatility of the real exchange rate in the baseline calibration as a function of the surplus consumption ratio in the home good and the foreign good. A large literature exists documenting stochastic volatility in exchange rates\textsuperscript{23} and one can see that the model is able to generate this feature endogenously. To understand the dynamics of the exchange rate volatility consider the diffusion coefficients

\[
\sigma_e(t) = \theta_H(t) - \theta_F(t).
\]  

(55)

From Equation (55) the diffusion coefficients of the real exchange rate is given by the difference in the Sharpe ratios in the home and the foreign country. In times when the surplus consumption ratios are low, the risk aversion is high and so are the Sharpe ratios. This implies that the volatility of the real exchange rate is high. Put differently, if the home country surplus consumption ratio is low, the home representative agent requires a high compensation for taking additional risk associated with the home good. Consequently the compensation for it (Sharpe ratio) must be high.

Table 5 shows a GARCH(1,1) model for the dollar-pound quarterly exchange rate series. To compare this to my model, I simulate 5000 quarters of exchange rate data. Table shows that both the data and the model produce highly persistent exchange rate volatility. The persistence is captured by the sum of the ARCH(1) and the GARCH(1) term. For the data this is 0.983 and for the model it is 0.987.

To shed further light on the relation between the surplus consumption ratios and the

\textsuperscript{22}See Obstfeld and Rogoff (2000).

\textsuperscript{23}See Poon and Granger (2003) for a review.
volatility of the real exchange rate I back out the surplus consumption ratios from the data. I assume that both the US and UK surplus consumption ratios are in their steady states at the beginning of the sample (1971 Q1). I then use the dynamics of the surplus consumption ratios and the realized shocks to GDP to back out the values of the surplus consumption ratios. Figure 4 shows the model implied and the realized real exchange rate. One can see from the figure that the model generates high volatility in the early 1980s and the early 1990s as in the data. However, the sharp decline in the real exchange rate volatility in the last part of the sample is not matched by the model. This might be partly due to a drop in output volatility in the later part of my sample, something my model does not capture.24 The correlation between the two series is 0.39. Next I regress the estimated real exchange rate standard deviation onto the model implied standard deviation as shown below

\[ \sigma_{Data,t+1} = \alpha + \beta \sigma_{Model,t} + \epsilon_{t+1}. \]  

The estimated slope coefficient, \( \beta \), is 0.81 with a standard deviation of 0.16. The constant, \( \alpha \), is \(-0.0060\) with a standard deviation of \(0.0228\). According to the model the slope coefficient should be one and the constant should be zero. Neither the null hypothesis that the slope is equal to one nor the intercept is equal to zero can be rejected. However, the joint hypothesis that both the slope is equal to one and the intercept is equal to zero is rejected. This is mainly due to the fact that the model implied average real exchange rate volatility is too high.

### 3.3 UIP Puzzle

Table 6 show the results from running UIP regressions on the US/UK data and data simulated from the benchmark model. The table shows results from nominal and real UIP

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24 The annual standard deviations of real GDP growth over the period 1971-1990 are 0.023 and 0.034 for US and UK respectively. For the period 1991-2008 the annual standard deviations are 0.011 and 0.015 for US and UK.
regressions for the data and real UIP regression for the model. The slope coefficient is negative for both real and nominal US/UK data. This is a well documented empirical regularity. Note that most empirical research on the failure of the UIP is conducted using nominal data. As my result illustrates the failure of the UIP is present in the real data as well. The slope coefficient for the data is \(-1.7835\) with a t-value of \(-1.8989\). The corresponding values for the model is \(-1.107\) with a t-value of \(-1.848\). As one can see the benchmark model can match the negative slope in the UIP regression. The explanatory power of the interest rate differential is small in the calibrated model compared to the data with \(R^2\) of 0 and 0.0392 respectively.

Figure 3 shows the interest rates, the interest rate differential and the excess return on the real exchange rate. For the baseline calibration the interest rate is high in times when the surplus consumption ratios are low. This contrasts with Verdelhan (2008) where pro-cyclical interest rates are necessary for matching the UIP puzzle. The interest differential is increasing in foreign surplus consumption ratio and decreasing in the home surplus consumption ratio. The interest rate differential is therefore pro-cyclical in the difference between the home and the foreign surplus consumption ratio. The counter-cyclical interest rate combined with pro-cyclical interest rate differential is driven by the difference between the elasticity of substitution and the risk aversion. While the risk aversion is important for the intertemporal smoothing motive for the interest rate, it is not important for the difference in the intertemporal smoothing motive between the home country and the foreign country. The excess return on the real exchange rate is decreasing in the home surplus consumption ratio and increasing in the foreign surplus consumption ratio. In times when the home good is close to the habit level, the representative agent is very risk averse to shocks to the home good. The exchange rate is highly correlated with shocks to the home good, and consequently the representative agent requires a large risk premium for holding exchange rate risk. Comparing the figure of the interest differential and the excess return on the exchange rate, one see that they move in opposite directions. The negative correlation between the interest differential
and the excess return on real exchange rate is necessary to match the failure of UIP. To reproduce the empirical stylized fact that high interest rate countries exhibit appreciating currencies, the volatility of the excess real exchange rate return must be higher than the interest rate differential. From the figure one can see that the baseline calibration is able to generate this feature. The range of the interest differential is small compared to the range of the exchange rate premium.

Figure 5 shows the risk free rate in the home country and the foreign country as I change the surplus consumption ratio in the home country for different values of the elasticity of substitution (top plots) and the risk aversion (bottom plots). All the other parameters are fixed as in the baseline calibration.\(^{25}\) From the figure one can see that both the elasticity of substitution and the risk aversion is important for the slope. In the top left and bottom left plot one can see that when the elasticity of substitution is low compared to the risk aversion, then the interest rate is pro-cyclical. This is because the precautionary savings motive becomes relatively more volatile compared to the intertemporal smoothing motive as one increase the risk aversion or lowers the elasticity of substitution. This illustrates that the model can potentially match both counter-cyclical and pro-cyclical real interest rates.\(^{26}\) Note that in the top plots when \(\beta = 0.2\) the home country interest rate is pro-cyclical in the home country surplus consumption ratio while the foreign interest rate is counter-cyclical.

Figure 6 shows the interest rate differential, the difference in the intertemporal smoothing motive and the difference in the precautionary savings motive as I change the surplus consumption ratio in the home country for different values of the elasticity of substitution (top plots) and the risk aversion (bottom plots). For the model to match the negative UIP slope the interest rate differential must be pro-cyclical. Two of the economies in the top

\(^{25}\) Since the calibration is symmetric in the two countries the plots will be similar if one instead change to the foreign surplus consumption ratio.

\(^{26}\) In Campbell and Cochrane (1999) the parameters are set such that the intertemporal smoothing motive exactly offsets the precautionary savings motive. One can see that in my model changing the parameters can change the slope. However, it is not possible to fix the parameters in such a way that these effect cancel each other out.
plots ($\beta = 0.2$ and $\beta = 0.4$) and two of the economies in the bottom plots ($\gamma = 4$ and $\gamma = 6$) satisfy this. For the other two economies the UIP slope will be greater than one. Note that the benchmark calibration is the only economy (out of the five economies in the plots) that satisfies counter-cyclical interest rates and pro-cyclical interest rate differential. The main mechanism for generating the pro-cyclical interest differential is a more volatile difference in precautionary savings motive than difference in intertemporal smoothing motive. The volatility of the difference in the precautionary savings motive is more sensitive to the risk aversion parameter, $\gamma$, than the difference in the intertemporal smoothing motive.

3.4 Carry Trades

Carry trades exploit the failure of the UIP by taking long positions in high interest rate countries and short positions in low interest rate countries. According to UIP, carry trades should not yield excess returns as the exchange rate for high interest rates countries should depreciate to offset the interest rate differential. However, from UIP regressions one can see that high interest rate countries tend to have an appreciating exchange rate, rather than depreciating. In Table 7, I calculate the summary statistics for carry trades using quarterly US and UK data from 1975 to 2008. The returns to carry trades are calculated as

$$r_{\text{carry}}(t + 1) = \begin{cases} 
\ln(e(t + 1)) - \ln(e(t)) + r_{UK}(t) - r_{US}(t) & \text{if } r_{UK}(t) > r_{US}(t) \\
\ln(e(t)) - \ln(e(t + 1)) + r_{US}(t) - r_{UK}(t) & \text{if } r_{UK}(t) < r_{US}(t)
\end{cases}$$

Carry trades are rebalanced every quarter. The Sharpe ratios are 0.699 and 0.417 for the real and nominal data respectively. Next I simulate data from my model and calculate the returns on carry trades. The exchange rate and interest rates are simulated on daily frequencies. I use the short rate to proxy for the 3 month interest rate as simulating the yields on 3 month bonds is computationally much more costly. Table 8 shows the results for different values of the risk aversion and the elasticity of substitution. The baseline calibration, $\gamma = 4$ and $\beta = 0.4$, has a Sharpe ratio of 0.092 for monthly frequency and 0.053 for quarterly
frequency. This is much lower than the corresponding values in the data. Increasing $\gamma$ to 6 and decreasing $\beta$ to 0.2 yield a Sharpe ratio closer to the data with values of 0.3 and 0.235 at monthly and quarterly frequencies respectively.

### 3.5 Dividend Leverage

From Table 4 one can see that the model implied volatilities of the stock markets are too low. In the benchmark calibration the stock markets are claims to aggregate outputs. In reality the dividends are more volatile than aggregate output. To accommodate more volatile dividends than output I include a leverage effect and pure dividend shocks:

$$
\frac{d\delta_i(t)}{\delta_i(t)} = \mu_i dt + \nu \sigma_i^\top dB(t) + \sigma_i^\top d\epsilon(t) \tag{58}
$$

where $\nu$ is leverage and $\epsilon = (\epsilon_1, \epsilon_2)$ is a two-dimensional Brownian motion. I assume that the agents can trade the claims to total output as in Equation (1), and that the stock market is the claim to total dividends as in Equation (58). This is to capture the fact that not all output is distributed as dividends. I set the target volatility of dividends to 12% as in Bansal and Shaliastovich (2008) and the leverage to 3. Dividend correlations are set to be 0.24 as in the benchmark calibration. The correlation between dividends and total output is 0.575. Table 9 shows the results of the calibration. The equity premium is above 6% for both the US and the UK. The model does a better job at matching the volatility of the stock markets with a standard deviation of 16.2% for the US and 19.3% for the UK compared to 15.5% for the US and 19.8% for the UK in the data. The equilibrium correlation is slightly higher than in the benchmark calibration with a value of 0.676. The remaining values are the same as in the benchmark calibration.
4 Conclusion

I study the effect of deep habits in a two country-two good pure exchange economy with consumption home bias in traded goods. Deep habits and consumption home bias allows for jointly matching the real exchange rate volatility, the failure of the uncovered interest rate parity, equity premiums, level and volatility of the real risk free rates and portfolio home bias. Habit formation at the country good level increase the volatility of the marginal utility relative to a model without habit formation, and thus helps matching the volatility of the real exchange rate and equity premiums. In times when the output of the home good is close to the home good habit level, then the risk premium for shocks to home output is high. The exchange rate is positively correlated with output shocks to the home good, and consequently the required risk premium on the exchange rate is high. I show that the interest rate differential is negatively correlated with the exchange rate premium if risk aversion is sufficiently high compared to the elasticity of substitution between the home and the foreign good. As exchange rate premiums are highly volatile due to habit formation, the model can generate a negative slope coefficient in UIP regressions. Although my model generates the failure of the UIP, the Sharpe ratio on a carry trade strategy in the model is too low compared to the data.
References


A Derivation of Equilibrium

In this section I derive equilibrium by using standard Martingale methods (see Cox and Huang (1989), Karatzas et al. (1990)). The Lagrangian of the central planner problem in Equation (19) is

$$L(Q^H, Q_F^H, Q^F_H, Q^F_F, y_H, y_F) = a u_H(Q^H_H, Q^H_F) + (1-a) u_F(Q^F_H, Q^F_F)$$

$$- y_H (Q^H_H + Q^H_F - Q_H) - y_F (Q^F_H + Q^F_H - Q_F).$$

(A.1)

The first order conditions are

$$a \frac{\partial u_H}{\partial Q^H_H} = y_1$$

(A.2)

$$a \frac{\partial u_H}{\partial Q^F_H} = y_2$$

(A.3)

$$(1-a) \frac{\partial u_H}{\partial Q^H_F} = y_1$$

(A.4)

$$(1-a) \frac{\partial u_H}{\partial Q^F_F} = y_2$$

(A.5)

$$Q^H_H + Q^F_H = Q_H$$

(A.6)

$$Q^H_F + Q^F_F = Q_F.$$  

(A.7)

Dividing Equation (A.3) by Equation (A.2) and Equation (A.5) by (A.4) yields

$$\left(\frac{1-\phi}{\phi}\right)^{1-\beta} \left(\frac{Q^H_H}{Q^F_F}\right)^{1-\beta} = \left(\frac{\phi}{1-\phi}\right)^{1-\beta} \left(\frac{Q^F_H}{Q^F_F}\right)^{1-\beta}.$$  

(A.8)

Inserting the market clearing conditions in Equation (A.6) and (A.7) into Equation (A.8) leads to

$$\left(\frac{1-\phi}{\phi}\right) \left(\frac{f^H_H}{1-f^H_H}\right) = \left(\frac{\phi}{1-\phi}\right) \left(\frac{f^F_F}{1-f^F_F}\right)$$

(A.9)

where $f^H_i = Q^H_i / Q_i$ is the habit adjusted consumption share of good $i$. Rearranging Equation (A.9) implies

$$f^F_F = \frac{B}{1-B}.$$  

(A.10)
where $B = \left( \frac{1 - \phi}{\phi} \right)^{2} \left( \frac{f_{H}^{H}}{1 - f_{H}^{H}} \right)$. The habit adjusted consumption share, $f_{H}^{H}$, solves

$$
\Gamma \left( (1 - \phi)^{1-\beta} (1 - f_{H}^{H})^{\beta} + \phi^{1-\beta} R^{\beta} \left( \frac{B}{1 - B} \right) \right)^{\frac{1 - \beta - \gamma}{\beta}} =
$$

(A.11)

$$
(1 - \phi)^{1-\beta} \Gamma \left( (1 - \phi)^{1-\beta} (1 - f_{H}^{H})^{\beta} + \phi^{1-\beta} R^{\beta} \left( \frac{B}{1 - B} \right) \right)^{\frac{1 - \beta - \gamma}{\beta}}
$$

where $R = Q_{F}/Q_{H}$ and $\Gamma = \frac{1 - a}{a} \left( \frac{1 - \phi}{\phi} \right)^{1-\beta}$. From the solution of the central planner problem in Equation (A.1) we have that the marginal utility of the representative agent is proportional to the home agent’s marginal utility evaluated at his optimal consumption, that is

$$
\nabla u(C_{H}(t), C_{F}(t), X_{H}(t), X_{H}(t)) = a \nabla u_{H}(C_{H}^{H}(t), C_{F}^{H}(t), X_{H}^{H}(t), X_{F}^{H}(t)).
$$

(A.12)

Consequently one can use the marginal utility of the home agent as the pricing kernel. I use the composite good of the home agent as numeraire, and consequently the state price density is proportional to the marginal utility of the home agent with respect to the composite good

$$
\frac{\xi_{H}(t)}{\xi_{H}(0)} = \frac{Z_{H}(t)^{-\gamma}}{Z_{H}(0)^{-\gamma}}.
$$

(A.13)

From the FOCs the relative prices are

$$
\frac{\partial u_{H}(C_{H}(t), X_{H}(t))}{\partial C_{i}^{H}} = P_{i}(t)\xi_{H}(t)
$$

(A.14)

for $i = H, F$. Using the optimal allocations yields

$$
P_{H}(t) = Z_{H}(t)^{1-\beta} \phi^{1-\beta} (Q_{H}^{H}(t))^{\beta - 1}
$$

$$
P_{F}(t) = Z_{H}(t)^{1-\beta} (1 - \phi)^{1-\beta} (Q_{F}^{H}(t))^{\beta - 1}.
$$

(A.15)

A.1 The Processes of the Habit Adjusted Consumption Allocations

Define the following quantity

$$
s_{i}^{\phi}(t) = \frac{\phi_{i}^{1-\beta} Q_{i}(t)^{\beta}}{\phi_{i}^{1-\beta} Q_{i}(t)^{\beta} + (1 - \phi_{i})^{1-\beta} Q_{i}(t)^{\beta}}.
$$

(A.16)
The following partial derivatives will be useful

\[
\frac{\partial Z_i}{\partial Q^i_H} = Z_i s_i^\phi (Q^i_H)^{-1} \tag{A.17}
\]

\[
\frac{\partial Z_i}{\partial Q^i_F} = Z_i \left(1 - s_i^\phi\right) (Q^i_F)^{-1} \tag{A.18}
\]

\[
\frac{\partial^2 Z_i}{\partial (Q^i_H)^2} = -Z_i (1 - \beta) s_i^\phi \left(1 - s_i^\phi\right) (Q^i_H)^{-2} \tag{A.19}
\]

\[
\frac{\partial^2 Z_i}{\partial (Q^i_F)^2} = -Z_i (1 - \beta) s_i^\phi \left(1 - s_i^\phi\right) (Q^i_F)^{-2} \tag{A.20}
\]

\[
\frac{\partial^2 Z_i}{\partial Q^i_H \partial Q^i_F} = Z_i (1 - \beta) s_i^\phi \left(1 - s_i^\phi\right) (Q^i_H)^{-1} (Q^i_F)^{-1} \tag{A.21}
\]

By Ito’s lemma we have the following

\[
dQ^i_H(t) = Q^i_H(t) \left(\mu_{Q^i_H}(t) dt + \sigma_{Q^i_H}(t)^\top dB(t)\right) \tag{A.22}
\]

\[
dQ^i_F(t) = Q^i_F(t) \left(\mu_{Q^i_F}(t) dt + \sigma_{Q^i_F}(t)^\top dB(t)\right) \tag{A.23}
\]

where

\[
Q^i_H(t) \mu_{Q^i_H}(t) = \frac{\partial g^i_H}{\partial Q^i_H} Q^i_H(t) \mu_{Q^i_H}(t) + \frac{\partial g^i_H}{\partial Q^i_F} Q^i_F(t) \mu_{Q^i_F}(t) \tag{A.24}
\]

\[
+ \frac{1}{2} \frac{\partial^2 g^i_H}{\partial (Q^i_H)^2} Q^i_H(t)^2 \sigma_{Q^i_H}(t)^\top \sigma_{Q^i_H}(t)
\]

\[
+ \frac{1}{2} \frac{\partial^2 g^i_H}{\partial (Q^i_F)^2} Q^i_F(t)^2 \sigma_{Q^i_F}(t)^\top \sigma_{Q^i_F}(t)
\]

\[
+ \frac{\partial^2 g^i_H}{\partial Q^i_H \partial Q^i_F} Q^i_H(t) Q^i_F(t) \sigma_{Q^i_H}(t)^\top \sigma_{Q^i_F}(t)
\]

\[
Q^i_F(t) \mu_{Q^i_F}(t) = \frac{\partial g^i_F}{\partial Q^i_H} Q^i_H(t) \mu_{Q^i_H}(t) + \frac{\partial g^i_F}{\partial Q^i_F} Q^i_F(t) \mu_{Q^i_F}(t) \tag{A.25}
\]

\[
+ \frac{1}{2} \frac{\partial^2 g^i_F}{\partial (Q^i_H)^2} Q^i_H(t)^2 \sigma_{Q^i_H}(t)^\top \sigma_{Q^i_H}(t)
\]

\[
+ \frac{1}{2} \frac{\partial^2 g^i_F}{\partial (Q^i_F)^2} Q^i_F(t)^2 \sigma_{Q^i_F}(t)^\top \sigma_{Q^i_F}(t)
\]

\[
+ \frac{\partial^2 g^i_F}{\partial Q^i_H \partial Q^i_F} Q^i_H(t) Q^i_F(t) \sigma_{Q^i_H}(t)^\top \sigma_{Q^i_F}(t)
\]
and

\[ Q_H^i(t)\sigma_{Q_H}(t) = \frac{\partial g_H^i}{\partial Q_H} Q_H(t)\sigma_{Q_H}(t) + \frac{\partial g_H^i}{\partial Q_F} Q_F(t)\sigma_{Q_F}(t) \]  
(A.26)

\[ Q_F^i(t)\sigma_{Q_F}(t) = \frac{\partial g_F^i}{\partial Q_H} Q_H(t)\sigma_{Q_H}(t) + \frac{\partial g_F^i}{\partial Q_F} Q_F(t)\sigma_{Q_F}(t). \]  
(A.27)

Next, the processes for the habit adjusted composite goods are

\[ dZ_i(t) = Z_i(t) \left( \mu_{Z_i}(t) dt + \sigma_{Z_i}(t)^\top dB(t) \right) \]  
(A.28)

where

\[ \mu_{Z_i}(t) = s_i^\phi(t) \mu_{Q_H}(t) + \left( 1 - s_i^\phi(t) \right) \mu_{Q_F}(t) \]
\[ -\frac{1}{2} (1 - \beta) s_i^\phi(t) \left( 1 - s_i^\phi(t) \right) \sigma_{Q_H}(t)^\top \sigma_{Q_H}(t) \]
\[ -\frac{1}{2} (1 - \beta) s_i^\phi(t) \left( 1 - s_i^\phi(t) \right) \sigma_{Q_F}(t)^\top \sigma_{Q_F}(t) \]
\[ + (1 - \beta) s_i^\phi(t) \left( 1 - s_i^\phi(t) \right) \sigma_{Q_H}(t)^\top \sigma_{Q_F}(t). \]  
(A.29)

### A.2 Proof of Proposition 1

This follows from application of Ito’s lemma on the state price density in Equations (42) and (43).

### A.3 Proof of Proposition 2

In complete markets the real exchange rate is given by the ratio of the foreign and the home country state price density. By Ito’s lemma

\[ d \left( \frac{\xi_F(t)}{\xi_H(t)} \right) = \frac{d\xi_F(t)}{\xi_H(t)} + \xi_F(t) d\left( \frac{1}{\xi_H(t)} \right) + d\xi_F(t) d\left( \frac{1}{\xi_H(t)} \right). \]  
(A.30)

Inserting the processes for the state price densities from Equation (2) and (3) into Equation (A.30) yields

\[ de(t) = e(t) \left( \mu_e(t) dt + \sigma_e(t)^\top dB(t) \right) \]  
(A.31)

where

\[ \mu_e(t) = r_H(t) - r_F(t) + \theta_H(t)^\top \sigma_e(t) \]  
(A.32)
and
\[ \sigma_e(t) = \theta_H(t) - \theta_F(t). \]  \hfill (A.33)

### A.4 Proof of Proposition 3

The stock prices are
\[ S_i(t)\xi(t) = E_t \left[ \int_t^T \xi(u)P_i(u)\delta_i(u)du \right]. \]  \hfill (A.34)

An application of Ito’s lemma to the left hand side of Equation (A.34) yields
\[ d(S_i(t)\xi(t)) = (S_i(t)\xi(t)) (\sigma_i(t) - \theta(t))^\top dB(t). \]  \hfill (A.35)

Apply Clark-Ocone theorem to the right hand side of Equation (A.34) yields
\[ dE_t \left[ \int_t^T \xi(u)P_i(u)\delta_i(u)du \right] = E_t \left[ \int_t^T D_t (\xi(u)P_i(u)\delta_i(u)) du \right]^\top dB(t). \]  \hfill (A.36)

Equating the diffusion coefficients in Equation (A.35) and (A.36) yields the result in Proposition 3.

### A.5 Proof of Proposition 4

The optimal portfolios are
\[ \pi_i(t) = (\sigma_S(t))^\top (W^i(t)\theta(t) + \frac{\Pi^i(t)}{\xi(t)}) \]  \hfill (A.37)

where
\[ W^i(t) = E_t \left[ \int_t^T \frac{\xi(u)}{\xi(t)} \left( P_H(u)C^i_H(u) + P_F(u)C^i_F(u) \right) du \right] \]  \hfill (A.38)

and where \( \Pi^i \) denotes the integrand in the martingale representation of \( W^i\xi \) (see Cox and Huang (1989)). Application of Clark-Ocone theorem on \( W^i\xi \) yields
\[ \Pi^i(t) = E_t \left[ \int_t^T D_t \left( \xi(u) (P_H(u)C^i_H(u) + P_F(u)C^i_F(u)) \right) du \right]. \]  \hfill (A.39)

By the product rule of Malliavin Calculus and from noting that \( C^i_H = Q^i_H + \phi_i X_H \) and \( C^i_F = Q^i_F + (1 - \phi_i) X_H \), we get the optimal portfolio policies in Proposition 4.
B Derivation of the Malliavin Derivatives

In this section I derive the Malliavin derivatives. The Malliavin derivatives of interest are $D_t \delta_i(u), D_t s_i(u), D_t X_i(u), D_t Q_i(u), D_t Q_i^j(u), D_t Z_j(u), D_t \xi_j(u)$ and $D_t P_i(u)$ for $u \geq t$ and $i \in \{H, F\}$. Note that each Malliavin derivative is a vector, where each element refers to the Malliavin derivative with respect to the first and second Brownian motion. The Malliavin derivatives of the dividend processes, $\delta_i$, are given by

$$D_t \delta_i(u) = \delta_i(u) \sigma_{\delta_i}. \quad (B.1)$$

To compute the Malliavin derivatives of the surplus consumption ratios, first note

$$ds_i(t) = \phi_i (\bar{s}_i - s_i(t)) dt + s_i(t) \lambda(s_i(t)) \sigma_{\delta_i}^T dB(t) \quad (B.2)$$

where $s_i(0) = s_i$ with $s_i \in (0, 1]$. The first variation process of the surplus consumption ratios are given by

$$Y_i(t) = \exp \left( - \int_0^t \left( \phi_i + \frac{1}{2} \sigma_{Y_i}^T(u) \sigma_{Y_i}(u) \right) du + \int_0^t \sigma_{Y_i}^T(u) dB(u) \right) \quad (B.3)$$

where

$$\sigma_{Y_i}(t) = \frac{\lambda(s_i(t)) (1 - 2 \lambda(s_i(t)))}{2(1 - s_i(t))} \sigma_{\delta_i}. \quad (B.4)$$

Using the relation between the first variation process and the Malliavin derivatives we get (see appendix C)

$$D_t s_i(u) = s_i(t) \lambda(s_i(t)) \sigma_{\delta_i} Y_i(u) Y_i(t)^{-1}. \quad (B.5)$$

The Malliavin derivative of $X_i(u)$ is

$$D_t X_i(u) = D_t ((1 - s_i(u)) \delta_i(u)) = \delta_i(u) D_t (1 - s_i(u)) + (1 - s_i(u)) D_t \delta_i(u). \quad (B.6)$$

Next I derive the Malliavin derivatives of $Q_i$

$$D_t Q_i(u) = D_t (\delta_i(u) s_i(u)) = \delta_i(u) D_t s_i(u) + s_i(u) D_t \delta_i(u) = Q_i(u) \sigma_{\delta_i} + \delta_i(u) s_i(t) \lambda(s_i(t)) \sigma_{\delta_i} Y_i(u) Y_i(t)^{-1}. \quad (B.7)$$
The Malliavin derivative of $Q^j$ follows from the chain-rule of Malliavin calculus

$$D_tQ^j_i(u) = \frac{\partial Q^j_i(u)}{\partial Q^H_i}D_tQ^H_i(u) + \frac{\partial Q^j_i(u)}{\partial Q^F_i}D_tQ^F_i(u). \quad (B.8)$$

The Malliavin derivative of $Z_j$ is

$$D_tZ_i(u) = Z_i(u)s^\phi_i(u)(Q^H_i(u))^{-1}D_tQ^H_i(u) + Z_i(u)\left(1 - s^\phi_i(u)\right)(Q^i_F(u))^{-1}D_tQ^F_i(u). \quad (B.9)$$

The Malliavin derivative of the state price density in country $j$ is

$$D_t\xi_j(u) = D_t\left((Z_j(u))^{-\gamma}\right) = -\gamma Z_j(u)^{-\gamma}D_t\ln(Z_j(u))$$

where $D_t\ln(Z_j(u)) = \frac{D_tZ_i(u)}{Z_j(u)}$. Finally, the Malliavin derivatives of the commodity prices are

$$D_tP_i(u) = (1 - \beta)P_i(u)\left(D_t\ln(Z^H_i(u)) - D_t\ln(Q^H_i(u))\right) \quad (B.11)$$

for $i = H, F$.

### C The First Variation Process and Malliavin Derivatives

In this section I will briefly discuss Malliavin calculus and the first variation process. Let $X^x(t)$ be an Itô process given by

$$dX^x(t) = \mu(X^x(t))dt + \sigma(X^x(t))^\top dB(t) \quad (D.1)$$

$$X^x(0) = x \quad (D.2)$$

where it is assumed that $\mu$ and $\sigma$ are in $C^1$ and satisfy standard condition such that there exists a unique strong solution to the SDE. Define the first variation process, $Y(t) = \frac{\partial}{\partial x}X^x(t)$, as follows

$$dY(t) = \mu'(X^x(t))Y(t)dt + \sigma'(X^x(t))^\top Y(t)dB(t) \quad (D.3)$$

i.e.

$$Y(t) = \exp \left( \int_0^t \mu'(X^x(u)) - \frac{1}{2} \sigma'(X^x(u))^\top \sigma'(X^x(u)) \, du + \int_0^t \sigma'(X^x(u))^\top dB(u) \right). \quad (D.4)$$
Now consider the Malliavin derivative of $X^x(t)$

$$Z(t) := D_s X^x(t) = \int_s^t \mu'(X^x(u)) D_s X^x(u) du + \int_s^t \sigma'(X^x(u)) D_s X^x(u) dB(u). \quad (D.5)$$

It then follows that

$$dZ(t) = \mu'(X^x(t)) Z(t) dt + \sigma'(X^x(t)) Z(t) dB(t)$$

$$Z(s) = \sigma(X^x(s)) \quad (D.6)$$

with the following solution

$$Z(t) = \sigma(X^x(s)) \exp \left( \int_0^t \left( \mu'(X^x(u)) - \frac{1}{2} \sigma'(X^x(u)) \sigma'(X^x(u)) \right) du \right). \quad (D.7)$$

Comparing with $Y(t)$ we see that

$$D_s X(t) = \sigma(X^x(s)) Y(t) Y(s)^{-1} \quad (D.8)$$

for $t \geq s$.

### D Computational Procedure

In this section I describe the numerical procedure to solve for the equilibrium quantities. The model is solved using Monte-Carlo (MC) simulations. The state variables are simulated forward using an Euler scheme with 20000 sample paths and 10000 time steps. I use antithetic sampling to reduce the variance. For each time step I calculate the optimal allocation of the habit adjusted consumption between the two agents by solving Equation (A.11) numerically by Newton’s method. The derivatives of of the optimal allocations $Q^i_j$ for $i, j = H, F$ are calculated using finite differences. The time integrals are calculated by using the trapezoid rule with 10000 steps.

### E Change of Numeraire

In this section I discuss how changing the numeraire good impacts the equilibrium. In the following I will work with the generic numerairs $A$ and $B$. The numeraire can be a single good or a basket of goods. The basket does not have to be a simple linear combination of the goods, but can take more complicated forms.
Proposition 5. The relation between the equilibrium Sharp Ratios under two different numeraires (A and B) is given by

$$\theta_B(t) = \theta_A(t) - \sigma_{p_A}(t).$$  \hspace{1cm} (F.1)

Furthermore, the relation between the risk free rates is given by

$$r_B(t) = r_A(t) - \mu_{p_A}(t) + \theta_A^T(t)\sigma_{p_A}(t).$$  \hspace{1cm} (F.2)

Stock price diffusion coefficients under two numeraires relate via

$$\sigma_B(t) = \sigma_A(t) - \sigma_{p_A}(t)$$  \hspace{1cm} (F.3)

where the subscript A (B) denotes the equilibrium quantities in the economy with A (B) as numeraire.
Table 1: **Summary Statistics.** The table shows the summary statistics of annualized real GDP growth and CPI growth over the period 1971-2008. The GDP data is from OECD and the CPI data is from IMF.

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>UK</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Output</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average GDP growth</td>
<td>0.024</td>
<td>0.024</td>
<td>0.024</td>
</tr>
<tr>
<td>Standard deviation GDP growth</td>
<td>0.019</td>
<td>0.027</td>
<td>0.023</td>
</tr>
<tr>
<td>ACF(1) GDP growth</td>
<td>0.319</td>
<td>-0.136</td>
<td>0.091</td>
</tr>
<tr>
<td>Correlation GDP growth</td>
<td></td>
<td></td>
<td>0.243</td>
</tr>
<tr>
<td><strong>Inflation</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Inflation</td>
<td>0.045</td>
<td>0.065</td>
<td>0.055</td>
</tr>
<tr>
<td>Standard deviation Inflation</td>
<td>0.016</td>
<td>0.030</td>
<td>0.023</td>
</tr>
<tr>
<td>ACF(1) Inflation</td>
<td>0.750</td>
<td>0.595</td>
<td>0.672</td>
</tr>
<tr>
<td>Correlation Inflation</td>
<td></td>
<td></td>
<td>0.653</td>
</tr>
</tbody>
</table>
Table 2: **Kalman Filter.** The table shows the estimates of Kalman filtering realized inflation. I estimate the following model with MLE

\[
\pi_{i,t+1}^i = \mu_{i,t}^i + \sigma_{\pi}^i \epsilon_{i,t+1}^i \tag{F.6}
\]

\[
\mu_{i,t+1}^i = \bar{\mu}_{\pi}^i + \phi_i^i \mu_{i,t}^i + \sigma_{\mu_{\pi}}^i \eta_{t+1}^i \tag{F.7}
\]

where \(\epsilon\) and \(\eta\) are i.i.d. standard normal. The table reports the estimated parameters \((\bar{\mu}_{\pi}^i, \phi_i^i, \sigma_{\mu_{\pi}}^i, \sigma_{\pi}^i)\) for \(i = US, UK\). \(\pi\) is the log growth of quarterly CPI. Data is from IMF and covers the period 1971Q1-2008Q1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>US</th>
<th>UK</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{\mu}_{\pi}^i)</td>
<td>0.00056</td>
<td>0.00110</td>
</tr>
<tr>
<td>(\phi_i^i)</td>
<td>0.94027</td>
<td>0.89823</td>
</tr>
<tr>
<td>(\sigma_{\mu_{\pi}}^i)</td>
<td>0.00218</td>
<td>0.00294</td>
</tr>
<tr>
<td>(\sigma_{\pi}^i)</td>
<td>0.00399</td>
<td>0.00916</td>
</tr>
</tbody>
</table>
Table 3: Model Parameters — Baseline Calibration. The table summarizes the model parameters for the baseline calibration. The calibration is symmetric in terms of the home good and the foreign good. The utility weights in the central planner problem are set to 0.5 for both the home agent and the foreign agent.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>Elasticity of substitution</td>
<td>$1/(1 - \beta)$</td>
</tr>
<tr>
<td>Time preference</td>
<td>$\rho$</td>
</tr>
<tr>
<td>Consumption home bias</td>
<td>$\phi$</td>
</tr>
<tr>
<td>Steady state habit level</td>
<td>$\bar{s}_i$</td>
</tr>
<tr>
<td>Speed of mean reversion habit</td>
<td>$\alpha_i$</td>
</tr>
<tr>
<td>Average output growth</td>
<td>$\mu_{\delta_i}$</td>
</tr>
<tr>
<td>Standard deviation output growth</td>
<td>$\sqrt{\sigma_{\delta_i}^2 \sigma_{\delta_i}}$</td>
</tr>
<tr>
<td>Cross-country correlation of consumption growth</td>
<td></td>
</tr>
</tbody>
</table>
Table 4: **Key Moments — Baseline Calibration.** The table shows the calibrated moments and the corresponding values in the data.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th></th>
<th>Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>US</td>
<td>UK</td>
<td>US</td>
<td>UK</td>
</tr>
<tr>
<td>Excess return</td>
<td>0.066</td>
<td>0.087</td>
<td>0.048</td>
<td>0.057</td>
</tr>
<tr>
<td>Average risk free rate</td>
<td>0.017</td>
<td>0.026</td>
<td>0.013</td>
<td>0.013</td>
</tr>
<tr>
<td>Standard deviation market</td>
<td>0.155</td>
<td>0.198</td>
<td>0.100</td>
<td>0.169</td>
</tr>
<tr>
<td>Standard deviation risk free rate</td>
<td>0.021</td>
<td>0.032</td>
<td>0.030</td>
<td>0.030</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.430</td>
<td>0.440</td>
<td>0.480</td>
<td>0.340</td>
</tr>
<tr>
<td>Correlation US and UK market</td>
<td>0.610</td>
<td></td>
<td></td>
<td>0.572</td>
</tr>
<tr>
<td>Standard deviation real exchange rate</td>
<td>0.110</td>
<td></td>
<td></td>
<td>0.136</td>
</tr>
<tr>
<td>Autocorrelation real exchange rate growth</td>
<td>-0.017</td>
<td></td>
<td></td>
<td>-0.017</td>
</tr>
<tr>
<td>Fraction of domestic stock held by domestic investors</td>
<td>-</td>
<td>-</td>
<td>0.885</td>
<td>0.885</td>
</tr>
</tbody>
</table>
Table 5: **GARCH(1,1)**. The table shows the coefficients and the t-values for a GARCH(1,1) estimation. The data column is estimated using quarterly real exchange rate data from 1971Q1-2008Q1. The model column is estimated on 5000 quarters of simulated data using parameters from the baseline calibration.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th></th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>t-stat</td>
<td>Coefficient</td>
</tr>
<tr>
<td>C</td>
<td>0.0000478</td>
<td>0.6218</td>
<td>5.42413E-05</td>
</tr>
<tr>
<td>ARCH(1)</td>
<td>0.0943196</td>
<td>1.3142</td>
<td>0.04443033</td>
</tr>
<tr>
<td>GARCH(1)</td>
<td>0.8887972</td>
<td>11.3145</td>
<td>0.942879036</td>
</tr>
<tr>
<td>ARCH(1)+GARCH(1)</td>
<td>0.9831168</td>
<td></td>
<td>0.987309366</td>
</tr>
</tbody>
</table>
Table 6: UIP regression. The table shows the coefficients and the Newey-West corrected t-values for quarterly UIP regressions. The data column is estimated using quarterly data from 1975-2008. The model column is estimated on 5000 quarters of simulated data using parameters from the baseline calibration.

<table>
<thead>
<tr>
<th>Data</th>
<th>Nominal</th>
<th>Model</th>
<th>Real</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>t-stat</td>
<td>Coefficient</td>
<td>t-stat</td>
<td>Coefficient</td>
</tr>
<tr>
<td>Constant</td>
<td>-0.0109</td>
<td>-1.3283</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Slope</td>
<td>-1.7278</td>
<td>-1.4193</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>R2</td>
<td>0.0232</td>
<td>-</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 7: **Carry trades - US and UK.** The table reports summary statistic on the carry trades using US and UK data over the period 1975-2008. Returns are calculated according to

\[
r_{\text{carry}}(t + 1) = \begin{cases} 
\ln(e(t + 1)) - \ln(e(t)) + r_{\text{UK}}(t) - r_{\text{US}}(t) & \text{if } r_{\text{UK}}(t) > r_{\text{US}}(t) \\
\ln(e(t)) - \ln(e(t + 1)) + r_{\text{US}}(t) - r_{\text{UK}}(t) & \text{if } r_{\text{UK}}(t) < r_{\text{US}}(t)
\end{cases}
\] (F.9)

<table>
<thead>
<tr>
<th>Data</th>
<th>Real</th>
<th>Nominal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation of the exchange rate growth</td>
<td>0.110</td>
<td>0.108</td>
</tr>
<tr>
<td>Correlation interest rate differential and exchange rate growth</td>
<td>-0.216</td>
<td>-0.175</td>
</tr>
<tr>
<td>Expected return carry trade</td>
<td>0.075</td>
<td>0.046</td>
</tr>
<tr>
<td>Standard deviation carry trade</td>
<td>0.108</td>
<td>0.109</td>
</tr>
<tr>
<td>Sharpe ratio carry trade</td>
<td>0.699</td>
<td>0.417</td>
</tr>
</tbody>
</table>
Table 8: **Carry Trades - Model.** The table reports model output for different value of risk aversion, $\gamma$, and the parameter governing the elasticity of substitution between the home and the foreign good, $\beta$. All other parameters are as in the benchmark calibration. I use the short rate to proxy for both the monthly and the quarterly real interest rate.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td><strong>Monthly</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation real exchange rate</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correlation interest rate differential and exchange rate growth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UIP slope</td>
<td>2.400</td>
<td>1.730</td>
<td>1.278</td>
</tr>
<tr>
<td>t-stat</td>
<td>2.321</td>
<td>2.609</td>
<td>2.769</td>
</tr>
<tr>
<td>R2</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>Expected return carry trade</td>
<td>-0.006</td>
<td>-0.006</td>
<td>-0.003</td>
</tr>
<tr>
<td>Standard deviation carry trade</td>
<td>0.177</td>
<td>0.137</td>
<td>0.092</td>
</tr>
<tr>
<td>Sharpe ratio carry trade</td>
<td>-0.035</td>
<td>-0.041</td>
<td>-0.035</td>
</tr>
<tr>
<td><strong>Quarterly</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation real exchange rate</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correlation interest rate differential and exchange rate growth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UIP slope</td>
<td>1.188</td>
<td>1.203</td>
<td>1.080</td>
</tr>
<tr>
<td>R2</td>
<td>0.002</td>
<td>0.003</td>
<td>0.004</td>
</tr>
<tr>
<td>Expected return carry trade</td>
<td>-0.005</td>
<td>-0.003</td>
<td>-0.002</td>
</tr>
<tr>
<td>Standard deviation carry trade</td>
<td>0.171</td>
<td>0.132</td>
<td>0.088</td>
</tr>
<tr>
<td>Sharpe ratio carry trade</td>
<td>-0.029</td>
<td>-0.021</td>
<td>-0.021</td>
</tr>
</tbody>
</table>

**Other Moments**

| | | | | | | | | | |
| real short rates | 0.069 | 0.071 | 0.074 | 0.008 | 0.013 | 0.019 | -0.112 | -0.103 | -0.094 |
| Expected excess return US | 0.018 | 0.018 | 0.018 | 0.047 | 0.048 | 0.049 | 0.107 | 0.098 | 0.098 |
| Expected excess return UK | 0.029 | 0.025 | 0.021 | 0.066 | 0.057 | 0.052 | 0.142 | 0.105 | 0.094 |
| Standard deviation stock market US | 0.073 | 0.074 | 0.075 | 0.103 | 0.100 | 0.101 | 0.159 | 0.137 | 0.134 |
| Standard deviation stock market UK | 0.165 | 0.136 | 0.105 | 0.218 | 0.169 | 0.130 | 0.279 | 0.197 | 0.151 |
| Correlation US and UK stock markets | 0.486 | 0.632 | 0.821 | 0.308 | 0.579 | 0.838 | 0.345 | 0.620 | 0.899 |
| Fraction of domestic stock held by domestic investors | 0.876 | 0.888 | 0.920 | 0.874 | 0.885 | 0.910 | 0.868 | 0.875 | 0.893 |
Table 9: **Key Moments — Calibration with leverage.** The table shows the calibrated moments and the corresponding values in the data. Parameters are as in the benchmark calibration. Dividend leverage is set to 3 for both US and UK. The standard deviation of dividend growth is 0.12 and the correlation between dividend and output growth is 0.575 in both US and UK. Correlation between US and UK dividend growth is 0.24.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>US</td>
<td>UK</td>
</tr>
<tr>
<td>Excess return</td>
<td>0.066</td>
<td>0.087</td>
</tr>
<tr>
<td>Average risk free rate</td>
<td>0.017</td>
<td>0.026</td>
</tr>
<tr>
<td>Standard deviation market</td>
<td>0.155</td>
<td>0.198</td>
</tr>
<tr>
<td>Standard deviation risk free rate</td>
<td>0.021</td>
<td>0.032</td>
</tr>
<tr>
<td>Correlation US and UK market</td>
<td>0.610</td>
<td></td>
</tr>
<tr>
<td>Standard deviation real exchange rate</td>
<td>0.110</td>
<td></td>
</tr>
<tr>
<td>Autocorrelation real exchange rate growth</td>
<td>-0.017</td>
<td></td>
</tr>
</tbody>
</table>
Figure 1: **Realized and Expected Inflation.** The figure shows the realized quarterly inflation together with the estimated expected inflation.
Figure 2: Standard Deviation of the Real Exchange Rate. The figure shows the standard deviation of the real exchange rate as a function of the surplus consumption ratio in the home good and the foreign good. The economy is parameterized as in the baseline calibration (see table 3).
Figure 3: The Risk Free Rates, Interest Rate Differential and the Excess Return on the Real Exchange Rate. The figure shows the risk free rate in the home country, the foreign country, the interest differential and the excess return on the real exchange rate as a function of the surplus consumption ratio in the home good and the foreign good.
Figure 4: Model Implied and Realized Real Exchange Rate Volatility. The figure shows the model implied and the realized real exchange rate volatility. The model implied real exchange rate volatility is estimated using the formula for the real exchange rate volatility and the estimated surplus consumption ratios.
Figure 5: Risk Free Rates. The figure shows the risk free rate in the home country and the foreign country as I change the surplus consumption ratio in the home country for different values of the elasticity of substitution (top plots) and the risk aversion (bottom plots). The other parameters are as in the baseline calibration.
Figure 6: **Interest Rate Differential.** The figure shows the interest rate differential, the difference in the intertemporal smoothing motive and the difference in the precautionary savings motive as I change the surplus consumption ratio in the home country for different values of the elasticity of substitution (top plots) and the risk aversion (bottom plots). The other parameters are as in the baseline calibration.