Optimal dynamic hedging in incomplete commodity futures markets

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Abstract

The main objective of this paper is to fill the gap in the literature by addressing, in a continuous-time context, the issue of using commodity futures as vehicles for hedging purposes. We derive optimal demands for commodity futures contracts by an unconstrained investor, who can freely trade on the underlying spot asset and on a discount bond. The investor, with a constant relative risk aversion, faces a dynamically incomplete market. A three-factor variant of the Schwartz (1997) model, incorporating the explanation of the convenience yield in terms of an embedded timing option, is used where the spot price, interest rates and the convenience yield are allowed to change randomly over time. However, the market price of risk is allowed to vary stochastically and assumed to be affine in the state variables for tractability. Our results show that, following the martingale route, optimal demand can be divided into a speculative term – the traditional mean variance component, and two terms hedging against unfavourable shifts in the opportunity set and exhibiting different behaviour. The first hedging term protects the investor against random fluctuation of the interest rate and translates into the covariance of the bond with that of the horizon of the investor, while the second hedging term is due to the stochastic behaviour of the (square) market price of risk and can be decomposed into three terms sharing fundamental properties with the classical Merton-Breeden components.

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1. Introduction

Futures markets have experienced dramatic growth, worldwide, of both trading volume and contracts written on a wide range of underlying assets. These features make it easier to use futures contracts as hedging instruments against unfavorable changes in the opportunity set, i.e. changes in state variables or factors describing the economic/financial environment. The growing activity of these markets has been accompanied, since the original normal backwardation of Keynes (1930) and Hicks (1939), by a substantial body of literature devoted to pricing and hedging with futures contracts. Hedging decisions are examined for an unconstrained investor who can freely trade the underlying spot asset.

In an intertemporal portfolio choice framework, Merton (1971, 1973) and Breeden (1979, 1984) derived optimal assets allocation for an unconstrained investor, who maximizes his expected lifetime utility function of consumption under the budget constraint. This demand encompasses the commonly referred Merton-Breeden hedging terms that reflect the investor’s wish to hedge against the random fluctuations of the opportunity set – i.e. the fluctuation of the state variables. As is well-known, the utility maximization approach implies, however, that the optimal demand includes an additional speculative position which depends on the investor’s risk aversion, as well as on the instantaneous expected excess return over the interest rate for spot assets and only expected return for assets marked to market such as futures price. Other theoretical models involved with dynamic asset allocation with futures contracts (see, among others, Ho, 1984; Stulz, 1984; Adler and Detemple, 1988a, b; Duffie and Jackson, 1990; Briys et al., 1990; Duffie and Richardson, 1991; Lioui et al., 1996; Lioui and Poncet, 2001) deal with a constraint utility maximizer investor – the so called Traditional Hedging Model. In the same economic environment, the investor’s optimal futures demand consists of three terms: a mean-variance speculative term, a Merton-Breeden hedging component and a pure hedge element related to the non-traded position.

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2 Interested readers could refer to Lioui and Poncet (2005).
Commodity futures contracts are contracts written on commodity assets. An abundant literature has been devoted to pricing commodity futures. The models developed explain the evolution of the futures prices through the random evolution of several relevant state variables. The stochastic processes of these variables are specified exogenously. The convenience yield turns out to be the crucial variable which constitutes one of the main differences between spot commodity prices and prices of financial assets. The recent sharp increase in commodity prices has revived the interest in commodity risk management. Derivatives securities or contingent claims, futures contracts in particular, are major tools used by investors for hedging in order to mitigate their exposure to changes in commodity prices. Surprisingly, while there are a number of models, above mentioned, dealing with futures hedging, to our best knowledge, the specific case of commodity futures contracts with a stochastic convenience yield has not yet been addressed in the literature. An exception is Hong (2001) who examined the impact of a stochastic convenience yield on optimal hedging and on the term structure of open interest, i.e., the total number of contracts outstanding. Since interest rates are assumed to be constant, futures prices are treated as forward prices and the margin account associated with the investor’s futures position is not explicitly modelled.

The main objective of this paper is to bridge the gap in the literature by addressing, in a continuous-time context, the issue of using commodity futures as vehicles for hedging purposes. In his presidential address, Schwartz (1997) developed a three-factor model in which the spot commodity price, the instantaneous interest rate and convenience yield are stochastic. One of the extensions proposed by Hilliard and Reis (1998) is to allow Schwartz’s model to be consistent with the initial term structure of interest rates. Another extension is to let the market price of risk to be an affine function of

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4 Brennan (1991) defines the convenience yield as “the flow of services accruing to the owner of the physical inventory, but not to the owner of a contract for future delivery”. Indeed, physical inventory provides some services such as the possibility of avoiding shortages of the spot commodity and thus to maintain the production process or even to benefit from a (anticipating) future price increase.

5 It is well known that, if interest rates are deterministic or constant, futures and forward prices are identical (see Cox et al., 1981).
the state variables - Casassus and Collin-Dufresne (2005). In this paper, a variant of the Hilliard and Reis-Casassus and Collin-Dufresne three-factor model is used as the benchmark. The main distinctive feature of our model consists in taking into account, in accordance to the theory of storage, the explanation of the convenience yield in terms of an embedded timing option bearing a risk uncorrelated with the primitive assets on top of the usual one. The optimal demand for commodity futures contracts is derived for an unconstrained investor, who is allowed to freely trade on the primitive assets - namely the underlying spot commodity and a discount bond, aims at maximising the expected constant relative risk aversion (CRRA) utility function\(^6\) of his (her) final wealth. Because of the two sources of risk of the convenience yield, the investor faces a dynamically incomplete market. The hedging problem is thus solved in an incomplete market framework by adopting the methodology pioneered by Pagès (1987) and subsequently generalized by He and Pearson (1991) and Karatzas et al. (1991). The proportions are nevertheless derived explicitly in an elegant way using a change of probability specific to (CRRA) utility function.

Relatively few papers deal with futures hedging in incomplete markets and especially for the case of the unconstrained investor. Breeden (1984) studied the allocational role of futures markets and derived the demand for futures contracts by an unconstrained investor when the futures contracts are written on the state variables and have instantaneous maturity. The other authors considered the case of a constraint investor. Our results show that the optimal demand for commodity futures by the unconstrained investor has a classical structure in the sense that it is composed of a speculative part and of a hedging term. However, our specific treatment of the mathematical problem allows us to differentiate the hedging term into two parts – see Lioui and Poncet (2001) and Munk and Sorensen (2004). The first one, involving the covariance of the bond with the one of the maturity that of the horizon, is due solely to the random fluctuation of the interest rates: in our specific affine Gaussian model it is the unique deterministic allocation term. It is also, in the same way as the speculative demand, strictly monotone in relative risk aversion – nevertheless strictly increasing because of its hedging property while the speculative term is strictly decreasing. However, contrary to

\(^6\) Note that optimal demands are also derived for the special cases of the Bernoulli and the infinitely risk-averse investor.
the mean variance term, it changes sign depending if the investor is more or less risk averse than the Bernoulli investor. The investor, who is less risk averse than the logarithmic one, shows thus reverse hedging behaviour as explained by Breeden (1984). The last hedging term is in accordance to Munk (2005) theorem 6.4 in the sense that it only involves the square market price – and the volatility of the bond of maturity that on the horizon. In our case, this term is stochastic and in particular affine in the state variables and can then easily be decomposed into three Merton-Breeden terms. However, our change of probability measure allows us to show that there exists a relative risk aversion for which this term has an optimum: two; a feature that was prior noticed numerically by Munk (2005). As a coincidence, it is also roughly the relative risk aversion measured by Meyer and Meyer (2004).

The remainder of the paper is organized as follows. In section 2, the economic framework is described and the investor’s optimization problem is formulated. Section 3 is devoted to the derivation of the optimal asset allocation for the unconstrained investor. An illustration of this demand is given in section 5. Section 6 offers some concluding remarks and suggests some potential future extensions. Proofs of propositions 1 and 2 are provided in the Appendix A and B.

2. The general economic framework

Consider a continuous-time frictionless economy. The uncertainty in the economy is represented by a complete probability space \((\Omega, F, P)\) with a standard filtration \(F = \{F(t) : t \in [0, T]\}\), a finite time period \([0, T]\), the historical probability measure \(P\) and a 4-dimensional vector of independent standard Brownian motions, \(z(t) = (z_s(t), z_u(t), z_f(t), z_c(t))\) - hereafter referred to as the orthogonal basis, defined on \((\Omega, F)\), where \(^\top\) stands for the transpose.

In this section, following Schwartz (1997), Hilliard and Reis (1998) and Casassus and Collin-Dufresne, three imperfectly correlated factors are assumed to be associated with the dynamics of the futures prices: the spot commodity price, \(S(t)\), the instantaneous forward rate, \(f(t,T)\), and the instantaneous convenience yield \(\delta(t)\).
The spot price of the commodity, \( S(t) \), satisfies the following stochastic differential equation (SDE hereafter):

\[
\frac{dS(t)}{S(t)} = (\mu_s(t) - \delta(t))dt + \sigma_s dz_s(t)
\]

(1)

with initial condition \( S(0) = S \). \( \mu_s(t) = r(t) + \lambda_s(S(t), \delta(t)) \) \( \sigma_s \) is the instantaneous expected rate of return of the spot price, \( \sigma_s \) represents the constant, strictly positive, instantaneous volatility of the spot price and \( \lambda_s(S(t), \delta(t)) = \lambda_s + \lambda_{s0} \delta(t) + \lambda_{sX} X(t) \), where \( X(t) = \ln(S(t)) \), stands for the market price of risk associated with the spot price process. To characterise the dependence of the latter on the level of inventories – Brennan (1958), we let it be an affine function of both the (log) of the spot price and the convenience yield. \( r(t) \) denotes the instantaneous riskless interest rate.

The instantaneous forward rate (hereafter the forward rate), \( f(t,T) \), at date \( t \), for delivery at time \( T, \ 0 \leq t \leq T \), is governed by the following stochastic process (see Heath et al., 1992):

\[
\begin{align*}
df(t,T) &= \mu(t,T)dt + \sigma_f(t,T)dz_f(t) \\
&= \mu(t,T)dt + \sigma_f(t,T)\left[\rho_{zf}dz_s(t) + \rho_{zf}dz_u(t)\right]
\end{align*}
\]

(2)

where \( f(0,T) \) is the initial forward yield curve, \( f(t,t)=r(t) \). \( \rho_{zf} \) is the instantaneous correlation and we call \( z^{cof}(t)' = (z_s(t), z_f(t), z_c(t), z_y(t)) \) the correlated basis – see below for the definition of \( z_c(t), z_y(t) \) and appendix A for the standard Cholesky decomposition that links the orthogonal basis to the correlated basis. \( \mu_f(t,T) \) and \( \sigma_f(t,T) \) represent respectively the drift and the strictly positive, instantaneous volatility of \( f(t, T) \). They satisfy the usual conditions (see Heath et al., 1992). In order to obtain analytical solutions, \( \sigma_f(t,T) \) is supposed to be deterministic and is restricted to the exponential case: \( \sigma_f(t,T) = \sigma_f e^{-\alpha(t-T)} \), where \( \sigma_f \) and \( \alpha \) are positive constants. As far as the spot interest rate is concerned, this model translates into an extended Vasicek model:

\[
\begin{align*}
dr(t) &= \alpha(\theta(t) - r(t))dt + \sigma_f\left[\rho_{zf}dz_s(t) + \rho_{zf}dz_u(t)\right] \\
\theta(t) &= f_0' + \frac{1}{\alpha} \frac{\partial f_0'}{\partial t} + \frac{\sigma_f^2}{2\alpha^2} \left[1 - e^{-2\alpha t}\right] + \frac{\sigma_f}{\alpha} \lambda_f(r(t))
\end{align*}
\]

(3) (4)
where $\lambda_{f}(r(t)) = \lambda_{f0} + \lambda_{fr}r(t)$ is the affine market price of risk associated with the interest rate.

The price, at time $t$, of a riskless discount bond with maturity $T_B$, $0 \leq t \leq T_B$, is equal to:

$$B(t, T_B) = \exp\left\{-\int_{t}^{T_B} f(t, s) ds\right\}$$

And, by applying Itô’s lemma, it obeys the following SDE:

$$\frac{dB(t, T_B)}{B(t, T_B)} = \mu_B(t, T_B)dt - \sigma_B(t, T_B)\left[\rho_{sf}dz_s(t) + \rho_{sz}dz_z(t)\right]$$

with initial condition $B(0, T_B)$. $\mu_B(t, T_B) = r(t) - \sigma_B(t, T_B)\lambda_{f}(r(t))$, is the bond expected return and its volatility is such that: $\sigma_B(t, T_B) = \int_{t}^{T_B} \sigma_f(t, s) ds = \sigma_fD_a(t, T_B)$, and $D_a(t, y) = \frac{1-e^{-x(y-t)}}{x}$.

A locally riskless asset, the savings account, is also available such that: $\beta(t) = \exp\left\{\int_{0}^{t} r(s) ds\right\}$, with initial condition $\beta(0)=1$.

The instantaneous convenience yield evolves stochastically over time following a mean-reverting process:

$$d\delta(t) = \kappa(\delta - \delta(t))dt + \sigma_c dZ_c + \sigma_y dZ_y$$

$$= \kappa(\delta - \delta(t))dt + \sigma_c \left[\rho_{sc}dZ_s(t) + \rho_{sz}dz_s(t)\right] + \sigma_y \left[\rho_{sy}dz_y(t) + \rho_{sz}dz_z(t)\right]$$

with initial condition $\delta(0)=\delta$. $\kappa$, $\delta$, $\sigma_c$ and $\sigma_y$ are positive constants. $\rho_{sc}, \rho_{sz}, \rho_{sy}, \rho_{sz}$ are specified in the Cholevsky decomposition in the appendix. The two markets price of risk, $\lambda_c(\delta(t)) = \lambda_{c0} + \lambda_{c3}\delta(t)$ and $\lambda_y(\delta(t)) = \lambda_{y0} + \lambda_{y3}\delta(t)$, are also time homogenous affine functions of the convenience yield so that the latter follows an autonomous process under both probabilities.

The convenience yield has a tendency to revert to a constant long-run interest level, $\delta$, with a speed of mean reversion $\kappa$. Empirical studies (see Fama and French 1988; Brennan 1991) found that the convenience yield should be specified by a mean-reverting process. The convenience yield is subject to two different kinds of random shocks. The first one, $z_c(t)$ is correlated to the two primitive assets – namely the commodity and the bond, however imperfectly. It is therefore related to
financial markets. The second one, \( z_y(t) \) is specific to the convenience yield and is uniquely related to real economy decisions. The Cholevsky decomposition shows that the convenience yield has two idiosyncratic sources of risks, \( z_y(t) \) and \( z_C(t) \). The first one arises because \( z_C(t) \) is only imperfectly correlated to primitive assets and the second one because of real economy decisions. We will see later that these two idiosyncratic sources of risks play a major role for financial decisions – i.e. asset allocation. The theory of storage (see Kaldor, 1939; Working 1948, 1949; Telser, 1958; Brennan, 1958; 1991) provides an explanation for the convenience yield to be couched in terms of an embedded timing option (see Litzenberger and Rabinowitz, 1995; Routledge et al., 2000). This property reflects the dual character of a commodity which can be viewed as a financial asset (storing a commodity now for future consumption) or as a consumption good (consuming the commodity immediately).

It is clear from the definition of our market that \( Y(t) = \begin{bmatrix} r(t) & \delta(t) & X(t) \end{bmatrix} \) is the right state variable vector that describes the economy. The price of a futures contract, at date \( t \), of maturity date \( T_H \), written on a commodity is then noted \( H(Y(t), t, T_H) = H(t, T_H) \). To determine the instantaneous return of the futures price, assume that this price function is twice continuously differentiable in \( Y(t) \).

The Feynman-Kac representation allows us to find a closed form solution for the futures price:

\[
H(t, T_H) = S(t) \exp \left\{ - \delta(t) D_x\phi(t, T_H) + r(t) D_a(t, T_H) + N(t, T_H) \right\} \kappa^0 = \kappa + \sigma_C \lambda_C + \sigma_y \lambda_y
\]

with the terminal condition \( D_x(t, T_H) = D_x(0, T_H) = N(T_H) = 0 \). Applying Itô's lemma and the marked to market condition to \( H(t, T_H) \) yields:

\[
\frac{dH(t, T_H)}{H(t, T_H)} = \mu_H(t, T_H) dt + \sigma_S dz_S(t) + \sigma_B(t, T_H) dz_f(t) - D_x\phi(t, T_H) \sigma_C dz_C(t) + \sigma_y dz_y(t)
\]

\[
\mu_H(t, T_H) = \sigma_S \lambda_S(S(t), \delta(t)) + \sigma_B(t, T_H) \lambda_f(r(t)) - D_x\phi(t, T_H) \sigma_C \lambda_C(\delta(t)) + \sigma_y \lambda_y(\delta(t))
\]

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7 The presence of \( z_y(t) \) differentiates the setting of our model from those of Schwartz (1997) and Hilliard and Reis (1998).
8 There is no need to specify the expression of \( N(t, T_H) \), since it will not be used in the rest of the paper; besides we are not interested in the pricing of futures contracts.
The futures price changes are credited or debited to a margin account with interest at the continuously compounded interest rate $r(t)$. The futures contract is indeed assumed to be marked to market continuously rather than on a daily basis, and then to have always a zero current value. The current value of the margin account, $X(t)$, is then equal to:

$$X(t) = \int_0^t \exp\left(\int_u^t r(v)dv\right) \theta_H(u, T_H) dH(u, T_H)$$

Applying Itô’s lemma to the above equation yields:

$$dX(t) = r(t)X(t)dt + \theta_H(t, T_H) dH(t, T_H)$$

where $\theta_H(t, T_H)$ represents the number of the futures contracts held at time $t$, such that $\theta_H(0, T_H) = 0$.

The market described above is dynamically incomplete. The commodity spot is a traded asset, the forward rate is a variable perfectly correlated with the bond, while the convenience yield is neither a traded asset nor a variable perfectly correlated with a traded asset. Since the futures price is a function of the convenience yield, the futures contract is then a non-redundant security. However, the convenience yield is assumed to have two specific stochastic components. Introducing a futures contract into the financial market does not allow to (dynamically) span the market: the number of sources of risk (Brownian motions) is strictly higher than the number of traded risky securities. The presence of an additional source of uncertainty associated with the convenience yield breaks down market completeness.

In a dynamically complete market, in absence of arbitrage opportunities (AAO), there exists a unique risk neutral probability measure (see Harrison and Pliska, 1981), $Q$, equivalent to the historical probability $P$, such that the relative price (with respect to the savings account chosen as numeraire), of any risky security is a $Q$-martingale. The essence of dynamically complete markets is the ability to exactly replicate the payoff of a derivative security written on traded assets by a dynamic portfolio strategy of these assets. The price, at any date $t$, of a contingent claim must be, in AAO, equal to the value of the portfolio strategy. The no-arbitrage pricing is based on the existence of this portfolio strategy allowing to perfectly hedge all the risks related to future fluctuations of the state variables. Karatzas et al. (1987) and Cox and Huang (1989; 1991) used the martingale approach to study the
consumption-portfolio problem with complete markets in a continuous-time setting. Their main idea is to transform this dynamic problem into a static one. The solution to the latter, in a first step, allows them to obtain the consumption and the terminal wealth as a function of the pricing kernel: \( \frac{1}{\beta(t)} \frac{dQ}{dP} |_{F_t} \). Then, optimal hedging demands are computed: In AAO, there exists an admissible trading strategy that replicates the consumption-final wealth pair forming a single budget constraint.

However, market completeness may be achieved through severe restrictions either on the market structure (market frictions and infrequent trading, for instance) or on the nature and the dynamics of the state variables. When markets are incomplete, some risks cannot be hedged, hence the “construction” of a portfolio of marketed securities that exactly replicates the payoff of a contingent claim is no longer always possible: there exist contingent claim which are not redundant. In this case, there exists an infinite number of equivalent martingale measures associated with a pricing kernel. The individual dynamic consumption and portfolio decision problem is more involved and cannot be solved by directly employing the same methodology as when markets are complete. Indeed, since there are infinitely many martingale measures, then there is an infinite number of budget constraints. However, He and Pearson (1991) tackled this problem by reducing this very number to a single budget constraint associated with one of the martingale measures, noted \( \tilde{P} \), the minimax (local) martingale measure in He and Pearson’s terminology: the minimax (local) martingale is the (local) martingale that minimizes among all the probability measure the static expected utility maximization programs. This can be accomplished by an appropriate completion of the market with fictitious risky assets (see Karatzas et al., 1991) whose instantaneous returns are orthogonal to those of the traded securities. The risk premia of the fictitious assets are, at optimum, determined in such a way that the investor does not invest in these assets. In other words, the agent minimizes his (her) maximized expected utility such that his (her) optimal demand for the fictitious assets is zero. With the appropriate completion of the market, the Karatzas et al. (1987) and the Cox and Huang (1989) approach applies and the optimal consumption-final wealth replicating strategy includes traded securities only.
The unconstrained investor has an investment horizon \( T_i \), \( 0 \leq t \leq T_i \leq T_H, T_B \), and he (she) is supposed to have a utility function that exhibits constant relative risk aversion equal to \( \gamma \), such that:

\[
U(W(t)) = \frac{W(t)^{1-\gamma}}{1-\gamma} \tag{5}
\]

where \( U(.) \) is a Von Neumann-Morgenstern utility function. When \( \gamma = 1 \), the “reference” utility in the finance literature is obtained, that is, the logarithmic utility function characterizing a Bernoulli investor:

\[
U(W(t)) = \ln W(t). \tag{5.1}
\]

In this case, the investor behaves myopically in such a way that his (her) hedging demand will not include any component associated with a stochastic opportunity set.

To determine the optimal assets allocation of commodity, bond and futures contract, each investor maximizes the expected utility function of his (her) terminal wealth. Under the martingale approach described above, the dynamic portfolio choice problem for each investor can be transformed into the following static one (Program \( \Pi \)):

\[
\max_{W(T_i), 0 \leq t \leq T_i} \mathbb{E} \left[ \frac{W(T_i)^{1-\gamma}}{1-\gamma} \bigg| F_t \right] \tag{6}
\]

s.t. \( W(t)h^\Lambda(t) = \mathbb{E}[W(T_i)h^\Lambda(T_i) \big| F_t] \)

where \( h^\Lambda(t) = \frac{\tilde{\xi}^\Lambda(t)}{\beta(t)} \) represents the pricing kernel – the inverse of the numeraire or optimal growth portfolio such that the value of any admissible portfolio relative to this numeraire is a martingale under \( P \) (see Long, 1990; Merton, 1990; Bajeux-Besnainou and Portait, 1997). \( \tilde{\xi}^\Lambda(t) \) is the Radon-Nikodym derivative of one of the probability measure \( \tilde{P} \) equivalent to \( P \) and depends on the parameter \( \Lambda \), the fictitious market price of risk. \( \tilde{\xi}^\Lambda(t) \), He and Pearson (1991) minimax martingale, which is one of this Radon-Nikodym derivative, will be determined at equilibrium - see below for a definition of \( \Lambda \) and the characterisation of \( \Lambda^* \).
3. Optimal asset allocation

Having described the framework for analysis, we will examine the optimal assets allocation problem for our unconstrained investor when the financial market is dynamically incomplete. The price dynamics of the risky traded securities writes in the orthogonal basis:

\[
\frac{dA(t)}{A(t)} = \left[ \frac{\sigma_S}{\sigma_B} \right] dt + \sigma d\tilde{z}(t)
\] (7)

\[
A(t) = \begin{bmatrix} S(t) & B(t) & H(t) \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_S^2 \\ \sigma_B^2 \\ \sigma_H^2 \end{bmatrix}
\]

is the (3 x 4)-dimensional volatility matrix which is assumed to be of full rank, where \( \sigma_{ij}, i \in \{S, B, H\} \) is the line volatility vector of each asset in the orthogonal basis. As shown by Pagès (1987) and by He and Pearson (1991), any market price of risk can be written in the form \( \Lambda = \lambda + \hat{\lambda} \) for some 4-dimensional adapted process. \( \lambda = \frac{\sigma^\dagger (\sigma \sigma^\dagger)^{-1}}{\sigma_H} \begin{bmatrix} \mu_S - r \\ \mu_B - r \\ -\mu_H \end{bmatrix} \) is the classic market price of risk although it differs slightly from the complete market case because \( \sigma \) is not invertible, while \( \hat{\lambda} \) is the unknown market price of risk that belongs to the kernel of \( \sigma \), i.e. \( \sigma \hat{\lambda} = 0 \). We show in the appendix B that \( \hat{\lambda} = 0 \). So \( \Lambda^* = \lambda \) and the minimax martingale measure then simply reads:

\[
\tilde{\mathbb{E}}^{\Lambda^*} = \exp \left\{ -\frac{1}{2} \int_0^t \| \lambda_s \|^2 - \int_0^t \lambda_s^\dagger d\tilde{z}(s) \right\}
\]

The martingale approach of Cox and Huang (1989) simply applies with the pricing kernel

\[
h^{\Lambda^*}(t) = \frac{\tilde{\mathbb{E}}^{\Lambda^*}(t)}{\beta(t)}
\]

and the optimal wealth follows:

\[
W(t) = t \left( \frac{1}{h^{\Lambda^*}(T_i)} \right)^{-\frac{1}{2}} \mathbb{E}_t \left( \left( \frac{h^{\Lambda^*}(T)}{h^{\Lambda^*}(T_i)} \right)^{-\frac{1}{2}} \right)
\]
Where \( \lambda \) is the lagrangian associated with the static program. This pricing kernel is artificial in the sense that, the market price of risk of the minimax measure \( \Lambda^* = \lambda \), differs from the natural market price of risk used to price derivatives assets. As a consequence, we note from on \( \Lambda^* = \lambda = \lambda_{\text{min/max}} \) the market price of risk that comes from the maximisation program and \( \Lambda^{\text{pricing}} \) the natural market price of risk in the orthogonal basis, used to price assets. The exact relation between them is linear and is derived in the appendix. It also sheds light on the idiosyncratic risk of the convenience yield.

As shown by Lioui and Poncet (2001), Munk and Sorensen (2005) and as it will be clearly seen later, the optimal wealth should be decomposed in the following way to exhibit the different nature of the hedging terms and to set aside the mean variance component:

\[
W(t) = \frac{1}{\gamma} h(t) \left[ \frac{1}{\gamma} B(t; T_i) \right]^{\frac{1}{\gamma}} E_t \left[ \frac{B(t; T_i) \beta(T_i) \tilde{\zeta}^{\Lambda^*}(t)}{B(T_i; T_i) \beta(T_i) \tilde{\zeta}^{\Lambda^*}(T_i)} \right]^{-\frac{1}{\gamma}}
\]

In the spirit of Jamshidian (1987, 1989) and Geman (1989, 1995) we operate a change of probability measure that is specific to CRRA utility functions – see Rodriguez (2002) for a one-dimensional example in a complete market setting:

\[
\frac{d\tilde{\mathcal{P}}^\gamma}{d\mathcal{P}} \bigg|_{F_t} = \exp \left[ -\frac{1}{2} \left( \frac{1}{\gamma} - 1 \right)^2 \int_0^T \| \lambda_{\text{min/max}}^{\text{min/max}} - \sigma_\theta(\theta; T_i) \| ^2 d\theta + \left( \frac{1}{\gamma} - 1 \right) \int_0^T \| \lambda_{\text{min/max}}^{\text{min/max}} - \sigma_\theta(\theta; T_i) \| ^2 d\theta \bigg] \right]
\]

Under the new probability \( \tilde{\mathcal{P}}^\gamma \), the integral in the conditional expectation of the last term is a classic finite variation integral to which the Feynman-Kac theorem directly applies:

\[
E_t \left[ \frac{B(t; T_i) \beta(T_i) \tilde{\zeta}^{\Lambda^*}(t)}{B(T_i; T_i) \beta(t) \tilde{\zeta}^{\Lambda^*}(T_i)} \right]^{-\frac{1}{\gamma}} = E_t^{\tilde{\mathcal{P}}^\gamma} \left[ \exp \left\{ \frac{1 - \gamma}{2\gamma^2} \int_t^T \| \lambda_{\text{min/max}}^{\text{min/max}} - \sigma_\theta(\theta; T_i) \| ^2 d\theta \right\} \right]
\]
For future reference, we let $E_t^{p_r} \left\{ \exp \left\{ \frac{1 - \gamma}{2\gamma^2} \int_t^T \sigma_{\theta}^{\text{max/min}} (\theta; T) d\theta \right\} \right\} = \exp \{ k(Y, \tau) \}$, where $\tau = T_i - t$. This convenient change of probability measure enable us to immediately notice that

$$
\frac{d}{d\gamma} \left\{ \frac{1 - \gamma}{2\gamma^2} \right\} (\gamma = 2) = 0 , \text{ so } k(Y, \tau) \text{ exhibits an optimum for } \gamma = 2 \text{ and so do } k_{ij}(Y, \tau).
$$

At any date $t$, the wealth of the investor is composed of $\theta_s(t)$, $\theta_B(t)$ and $\theta_\beta(t)$ units of respectively the spot commodity, the discount bonds and the riskless asset, and the margin account:

$$
W(t) = \theta_s(t) S(t) + \theta_B(t) B(t, T_B) + \theta_\beta(t) \beta(t) + X(t)
$$

(8)

Applying Itô’s lemma to expression (8), the dynamics of the unconstrained investor’s wealth constraint may be written:

$$
\frac{dW(t)}{W(t)} = \left[ r(t) + \pi' \sigma \lambda^{\text{pricing}} \right] dt + \pi' \sigma dz(t)
$$

(9)

$$
\pi' = [\pi_s(t) \quad \pi_B(t) \quad \pi_\beta(t)]. \quad \pi_s(t) = \frac{\theta_s(t) S(t)}{W(t)} , \quad \pi_B(t) = \frac{\theta_B(t) B(t, T_B)}{W(t)} \quad \text{ and }
$$

$$
\pi_\beta(t, T_H) = \frac{\theta_\beta(t) H(t, T_H)}{W(t)} \quad \text{denote, respectively, the proportions of the total wealth invested in the commodity, the discount bond and the futures contract.}
$$

**Proposition 1.** Given the economic framework described above, the optimal demand for risky assets by the unconstrained investor reads:

$$
\pi = \frac{1}{\gamma} \left( \sigma \sigma' \right)^{-1} \sigma \lambda^{\text{min/max}} (Y) + \left[ 1 - \frac{1}{\gamma} \right] \left( \sigma \sigma' \right)^{-1} \sigma \gamma (T_i) \gamma^r
$$

The optimal asset allocation can be decomposed in:

a) a traditional tangent component

$$
\pi^{\text{mv}} = \frac{1}{\gamma} \left( \sigma \sigma' \right)^{-1} \sigma \lambda (t) = \frac{1}{\gamma} \left( \sigma \sigma' \right)^{-1} \mu
$$
b) an hedging component related to the movement of the interest rate

\[
\pi_{Hedge\_rate} = \left[1 - \frac{1}{\gamma}\right] \left(\sigma \sigma'\right)^{-1} \sigma_b(T_f)
\]

c) an hedging component related to the movement of the (square) market price of risk

\[
\pi_{Hedge\_MPR} = \left(\sigma \sigma'\right)^{-1} \sigma_Y k(T_f, Y)
\]

**Proof.** See appendix B.

For the sake of interpretation we need to introduce two assets that can be duplicated even in our incomplete market framework by a portfolio of our four assets - namely the risk less asset, discount bond, commodity and futures contracts. Both of them reflect idiosyncratic risks. The first one is associated with the idiosyncratic risk of the interest rate while the second one to that of the convenience yield – the commodity plays the role of the third asset that spans the risk of \(dz(t)\). Note that the futures contract is essential in the making of the second synthetic asset:

\[
\frac{dB_u(t, T_B)}{B_u(t, T_B)} = \mu_B(t, T_B)dt - \sqrt{1 - \rho_\gamma^2} \sigma_B(t, T_B)dz(t)
\]

\[
= \mu_B(t, T_B)dt + \sigma_B(t, T_B)dz(t)
\]

\[
\frac{dH_{v,c}(t, T_H)}{H_{v,c}(t, T_H)} = \mu_{Hv,c}(t, T_H)dt - D_\varepsilon(t, T_H) \left[\sigma_{\varepsilon} \rho_{\varepsilon} + \sigma_\gamma \rho_\gamma \right]dz(t) + \sigma_\gamma dz(t)
\]

\[
= \mu_{Hv,c}(t, T_H)dt + \sigma_{Hv,c}(t, T_H)dz(t)
\]

We arbitrarily assume that they are standard cash assets – i.e. they are not marked to market. We then have \(\mu_{B_u}(t, T_B) = r(t) + \sigma_{B_u}(t, T_B)\Delta^{pricing}(Y)\) and \(\mu_{Hv,c}(t, T_H) = r(t) + \sigma_{Hv,c}(t, T_H)\Delta^{pricing}(Y)\).

**Proposition 2** The optimal mean-variance proportions can be couched in a recursive way:

\[
\pi_{MV}^H(t) = \frac{1}{\gamma} \left(\frac{\mu_{Hv,c}(t, T_H) - r(t)}{\sigma_{Hv,c}}\right)
\]

\[
\pi_{MV}^B(t) = \frac{1}{\gamma} \left[\frac{\mu_{B_u}(t, T_B) - r(t)}{\sigma_{B_u}} - \frac{\text{Cov}[B_u, H]}{\sigma_{B_u}^2} \pi_{MV}^H(t)\right]
\]
\[
\pi^m_S(t) = \frac{1}{\gamma} \left[ \frac{\mu_s(t) - r(t)}{\sigma_s^2} - \frac{\text{Cov}[S,H]}{\sigma_s^2} \pi^m_H(t) - \frac{\text{Cov}[S,B]}{\sigma_s^2} \pi^m_B(t) \right]
\]

Proof see appendix B.

Where \( \text{Cov}[M,N] \) \( M,N \in \{H,B_u,B,S\} \) designates the instantaneous covariance of the return of the assets.

This decomposition sheds light on the importance of the idiosyncratic risks represented by the two synthetic assets and, in the same spirit, the commodity. The futures contract is used to mimic the mean-variance efficient component linked to the idiosyncratic risk of the convenience yield multiplied by the usual risk tolerance – constant in our study case. However, when the investor buys a futures contracts he (she) also interferes with the idiosyncratic risk of the interest rate because of the correlation of the futures contract to the latter. The optimal demand in bond is then perturbed and is not only the mean-variance efficient component linked to \( B_u(t,T_B) \) - the idiosyncratic asset spanning the interest rate, but is diminished by the optimal demand in futures contracts up to a covariance factor of \( B_u(t,T_B) \) with \( H(t,T_H) \) that replicates the correlation. Note that if the synthetics assets were readily available to the unconstrained investor, we would not have this correcting term since \( \text{Cov}[B_u,H,v,c] = 0 \). The same logic applies to the optimal demand in commodities while it is diminished by two correcting terms because of both its correlation with \( H(t,T_H) \) and \( B(t,T_B) \). These equations also provide some intuition about the behaviour of the proportions. We know as an empirical fact that interest rates are barely correlated with the convenience yield and the commodity: this implies that the demand in bond will be mainly independent from the demand in futures contract and that the demand in commodities will be largely driven by the demand in futures contracts. And because the demand in commodities is related to the demand in futures contracts by a \( -\frac{\text{Cov}[S,H]}{\sigma_s^2} \) factor that is closed to -1, one can expect that its behaviour will show some opposite direction to that of the futures contract.
This formulation may also be convenient for computational purpose because it allows optimal allocation to be calculated in a recursive way: the demand in futures contracts is first derived, thereafter the one of the bond that is a function of the demand in futures contracts and finally that of the commodity, function of the two latter. Because of the affine structure of the market price of risk, the proportions are also affine in the state variables. Note that because the optimal proportion in futures contracts depend on the three state variables – see the proof in the appendix A, so do the optimal allocation in bond and commodity. Finally, this term is nil for an infinitely risk averse investor, which reflects its speculative aspect.

**Proposition 3** The optimal hedging proportions spawned by the interest rate is only carried by the bond and writes:

\[
\pi_{H}^{\text{Hedge\_rate}} = \pi_{B}^{\text{Hedge\_rate}} = 0
\]

\[
\pi_{B}^{\text{Hedge\_rate}}(t) = \left[1 - \frac{1}{\gamma}\right] \frac{\text{Cov}[B(T_t), B(T_B)]}{\sigma_B^2}
\]

Proof see appendix B.

This term is now usual – see for example Lioui and Poncet (2001) and Munk and Sorensen (2004), when one follows the martingale route. It is the sole of the three terms that is deterministic. One could think that this feature is intimately linked to the Gaussianity of our model. However, this characteristic would remain true if we would have used a CIR process for the interest rate instead of our HJM one – simply because the volatility of the bond shows a multiplicative independence between its time to maturity and the interest rate. This term is also strictly increasing in the relative risk aversion of the investor and is then maximum for the infinitely risk averse investor – in particular, this term is also the only one that is not nil, it is even maximum, for the infinitely risk averse investor and in that sense is a true hedging term. While it may not strictly be a Merton-Breeden term, it shares at least two of its fundamental properties: it is nil for the logarithmic investor and tends to zero as the horizon shrinks, reflecting the classic behaviour: "short term investors do not hedge". This term is negative for
investors with unbounded utility function, i.e. $\gamma < 1$, exhibiting so called reverse hedging – see Breeden (1984).

**Proposition 4** The optimal hedging proportions generated by the (square) market price of risk can be decomposed into three Merton-Bredeen like components- one for each state variable:

$$\pi_{Hedge_{MPR}} = \pi_{Hedge_{MPR} - r} + \pi_{Hedge_{MPR} - \delta} + \pi_{Hedge_{MPR} - X}$$

$$\pi_{Hedge_{MPR} - r} = (\sigma \sigma')^{-1} \sigma \sigma_{rY} k_r(t,Y),$$

$$k_r(t,Y) = \beta_{r}(T_{t} - t) + \gamma_{rY}(T_{t} - t)r + \gamma_{r\delta}(T_{t} - t)\delta + \gamma_{rX}(T_{t} - t)X$$

$$\pi_{Hedge_{MPR} - \delta} = (\sigma \sigma')^{-1} \sigma \sigma_{\deltaY} k_\delta(t,Y),$$

$$k_\delta(t,Y) = \beta_{\delta}(T_{t} - t) + \gamma_{\deltaY}(T_{t} - t)r + \gamma_{\delta\delta}(T_{t} - t)\delta + \gamma_{\deltaX}(T_{t} - t)X$$

$$\pi_{Hedge_{MPR} - X} = (\sigma \sigma')^{-1} \sigma \sigma_{XY} k_X(t,Y),$$

$$k_X(t,Y) = \beta_{X}(T_{t} - t) + \gamma_{XY}(T_{t} - t)r + \gamma_{X\delta}(T_{t} - t)\delta + \gamma_{XX}(T_{t} - t)X$$

Proof see appendix B.

Where $\beta_{i}(\tau_{i}), \gamma_{ij}(\tau_{i}) i, j \in \{r, \delta, X\}$ are deterministic functions of time nil at the origin.

The same remarks as above apply in the sense that each of the three terms is nil for Bernouilli investors and tends to zero when the horizon shrinks. They also change sign depending if $\gamma < 1$ or $\gamma > 1$. However, the components are not monotone in the relative risk aversion and exhibits an optimum for $\gamma = 2$. They are nil for an infinitely risk averse investor and in that sense are not truly hedging proportions. Finally, each $\sigma_{i}^{-1} k_i(t,Y), i \in \{r, \delta, X\}$ is homogenous to a market price of risk and is also affine in the state variables. However, this affine structure is time inhomogeneous because of the horizon effect.
Proposition 5 Each of the Merton-Breeden like component can also be decomposed in a recursive way similar to that of the mean variance component:

\[ \pi_H^{\text{Hedge}_\delta}(t) = \frac{\text{Cov}(\delta, H_v)_{\varepsilon}}{\sigma_H^2} k_\delta(t_i) \]

\[ \pi_B^{\text{Hedge}_\delta}(t) = \frac{\text{Cov}(\delta, B_u)_{\varepsilon}}{\sigma_B^2} k_\delta(t_i) - \frac{\text{Cov}(H, B)_{\varepsilon}}{\sigma_B^2} \pi_H^{\text{Hedge}_\delta}(t) \]

\[ \pi_S^{\text{Hedge}_\delta}(t) = \frac{\text{Cov}(\delta, S)_{\varepsilon}}{\sigma_S^2} k_\delta(t_i) - \frac{\text{Cov}(H, S)_{\varepsilon}}{\sigma_S^2} \pi_H^{\text{Hedge}_\delta}(t) - \frac{\text{Cov}(B, S)_{\varepsilon}}{\sigma_S^2} \pi_B^{\text{Hedge}_\delta}(t) \]

Proof see appendix B.

Where \( \text{Cov}(V, N), V \in \{r, \delta, X\}_{\varepsilon}, N \in \{H_v, B_u, S\} \) is the instantaneous covariance between the state variable and the return of the asset.

We have only shown the result for \( \pi_H^{\text{Hedge}_\delta} \). The same methodology applies for the two other terms by letting \( r(t) \) or \( X(t) \) take the place of \( \delta(t) \). However, their structure is reduced:

\[ \pi_H^{\text{Hedge}_\delta}(t) = 0 \] because the interest rate is not correlated to \( H_v_{\varepsilon} \) and

\[ \pi_H^{\text{Hedge}_\delta}(t) = \pi_B^{\text{Hedge}_\delta}(t) = 0 \] because the spot price is neither correlated with \( H_v_{\varepsilon} \) nor with \( B_u \).

This formulation reminds the structure of the Merton-Breeden hedging portfolio by exhibiting the covariance of the state variable with the assets. The amounts the investor should invest in these portfolios are affine in the state variables – as already said, the \( k_i(t_i), i \in \{r, \delta, X\} \) are time inhomogeneous affine functions of the state variables, and so we can intuit that the

\[ \frac{J_{WV}}{W_{J_{WW}}}, V \in \{r, \delta, X\} \] are also time inhomogeneous affine functions of the state variables where \( J \) is the classic indirect utility function of the dynamic programming approach.
The right assets the investor has to take into account are once again the synthetic ones and the proportions are perturbed in the same way as the mean-variance ones and all the remarks related to this phenomenon apply. Note again that each of these term is nil for the Bernoulli and tends to zero when the horizon shrinks. It is also nil for the infinitely risk averse investor, which confirms that the word “hedging” is not appropriate. Finally, the sign of the proportions are opposite for investors with unbounded utility functions ($\gamma < 1$) and investors with bounded utility functions ($\gamma > 1$).

4. An illustrative example

To get more insights on the impact of the parameters on the model, various simulations are provided in figures 1 to 17. We simulate each of the three groups of proportions as a function of the horizon and as a function of the state variables.

The parameter values are partly inspired from the Schwartz (1997) and Casassus and Collin-Dufresne (2005) model. We choose a positive value for the market price of risk of the commodity, that is, the commodity is expected to have a positive excess return equals to $\sigma S \lambda = 17.5\%$. We also choose a positive value for that of the convenience yield: the contango effect on the future contract linked to $\lambda S$ being positive is then mitigated by the backwardation effect of $\delta S$. The market price of risk of the interest rate is naturally chosen negative: the excess premium on the bond is positive. The coefficients of the market price of risk linked to the state variables are chosen so that they have opposite sign to their static component, exhibiting a kind of mean reversion effect. For example, the static component of the market price of risk of the commodity is positive, $\lambda_{50} = 0.7$, while the dynamic components are negative, $\lambda_{55} \delta = \lambda_{55} X = -0.1$, so that the total market price of risk is $\lambda S = 0.5$. 

[INSERT TABLE 1 ABOUT HERE]
For the set of figures where we draw the proportions as a function of the horizon, we let the investment horizon varies from $T_t \in [0;1.5]$ and for each horizon date, we arbitrary set the maturity of the futures contract and the bond such as $T_{mu} = T_t + 2/12$ and $T_{mb} = T_t + 5$. That is the investor buys a futures contracts that matures two months after the end of the investment and a bond five years after.

For each figure, the optimal demands are derived for four degrees of relative risk aversion (RRA). The first one is that of the investor that has unbounded from above utility function. As noted by Kim and Omberg (1996), the indirect utility function can explode for two low values of $\gamma$ under one. To avoid that problem, we choose a parameter close to one $\gamma = 0.8$. The second risk aversion parameter is the traditional logarithmic function, $\gamma = 1$, that separates bounded from unbounded utility functions. We also take $\gamma = 2$ as noted by Meyer and Meyer (2004) and to verify numerically that this value is an optimum for the hedging term spawned by the (square) market price of risk. Finally according to Mehra and Prescott (1985) risk aversion should be higher than one. To reflect this feature, we choose $\gamma = 6$.

While proportions reflect the choice of the parameters, and especially the magnitude and the signs of the market price of risk, one can nevertheless draw general facts. The mean variance demand in futures contract exhibits an opposite behaviour to the one of commodity as already announced. Both demands increases or decreases sharply as the horizon shrinks: this is due to the pattern of the volatility futures contract that flattens for long horizon but is highly non-linear when the horizon shrinks – remember that the time to maturity of the futures contracts is only two months longer than that of the horizon. The low correlation between the interest rate and both the convenience yield and the commodity imply that the volatility of the future
contract has little effect on the demand for bond. Given the choice of our parameters, the volatility of our bond slowly varies with the horizon, and so do the demand for bond.

[INSERT FIGURES 8, 9, 10 ABOUT HERE]

The hedging term due to the interest rate follows the pattern announced: increasing in relative risk aversion, zero at the origin, and exhibiting reverse hedging for the insatiable investor ($\gamma = 0.8$). We see however that it decreases (increases) sharply, almost linearly, with the time to horizon in a similar way to the hedging demand in bond due to the (square) market price of risk. However, both demands in bonds have opposite in sign confirming the fact that the component spawned by the (square) market price of risk is not really a hedging one. The terms due to the (square) market price of risk exhibit clearly a maximum (minimum) for the investor with $\gamma = 2$. Demands in commodity and futures contracts for the market price of risk components also follow a similar pattern but are opposite in sign. Contrary to the demand in bond, they are flat for long horizon and are again highly non-linear near zero. Our numerical example shows that they have the same sign as the speculative component but it should not be taken as a general fact. Nevertheless, this confirms that the term “hedging” is misleading.

[INSERT FIGURES 11, 12, 13 ABOUT HERE]

We now study the impact of a change in the value of the state variables. We take the value of the parameters in table one and arbitrary set the horizon investment to $T_f = 0.5$. We focus on one of the main feature of the paper and study mainly the impact of the convenience yield on the various demands – it is also the variable that has the most impact on allocation: this should come as no surprise considering our model. We let the convenience yield vary between $-5\%$ and $+20\%$. The effect of the variable is of course linear for all demands and the same remarks as above apply: the effect of the convenience yield is opposite for the demand in commodity
and in futures contract – both for the speculative and the market price of risk components. The impact of the convenience yield is strong on the bond demand even though the latter is poorly correlated with interest rates. The demand in commodity is decreasing with the convenience yield confirming the mean reverting pattern of the excess return. However, any intuitive interpretation is hazardous, because, as we saw in the above propositions, the right parameters to take into account are those linked to the idiosyncratic risks of the orthogonal basis. We can indeed only get an economic insight when we look at the parameters values in the correlated basis, which is linked to the orthogonal basis by a four dimensions Cholevsky transformation. And to interpret that four dimensions transformation is far beyond any human capability.

As a general rule, the interest rate as little impact in our model except for the speculative demand in bond. A rise in the interest rate is followed by a decrease in the optimal allocation in bonds which is in accordance to the mean reverting intuition. However, we also show the mean variance demand as a function of the log the spot price, and, although we allowed for the same mean reverting effect as the convenience yield and these two variables are positively correlated, its effect is exactly opposite to the one of the convenience yield, putting an end to any quick intuitive interpretation.

5. Concluding remarks

In this paper, optimal hedging decisions involving commodity futures contracts have been studied in a continuous-time context (i) for an unconstrained investor with a constant relative risk aversion utility function, (ii) when spot prices, interest rates and, especially, the convenience yield evolve randomly over time, and market prices of risk are stochastic (iii) when markets are incomplete. In this setting, we provided explicit solutions to the optimal assets allocation problem with an insight in futures contracts. We were able to confirm in an incomplete market setting distinctive features that one
should obtain when one follows the elegant and modern way of the martingale route. The demand is
divided in a speculative component and two so called hedging term. The first hedging term is a true
hedging term – maximum for the infinite risk averse investor, genuinely deterministic, and involves the
bond of maturity that of the investment. The second hedging originating from the (square) market price
of risk term is not a real one in the sense that it is nil for infinite risk averse investors, exhibits an
optimum for $\gamma = 2$ and is an affine time inhomogeneous (horizon effect) in the state variables. Both
the speculative component and the second hedging term can be couched in a recursive way that exhibits
the fundamental role of the idiosyncratic risks. However, even in our incomplete market framework,
one can synthesise two assets that exactly span these idiosyncratic risks – the futures contracts turns out
to be the fundamental asset used to span the convenience yield risk, which shows the intimate relation
between these two. These synthetic assets also shed light on how to interpret optimal demand and show
that any direct intuitive interpretation is heroic because of the complexity of the four dimensions
Cholevsky transformation needed for these assets.

The economic framework of this paper can be extended in several directions. Finally, it is now
acknowledged in the relevant litterature that the convenience yield is not observable: indeed, in a
partially observable economy (see, for instance, Dothan and Feldman 1986; Detemple 1986; Gennotte
1986; Brennan 1998; Xia 2001) an agent can estimate one or more unobserved state variable(s) given
information conveyed by past observations spawned by observable state variables via the continuous-
time Kalman-Bucy filter. One important extension would therefore be to study the effect of incomplete
information and, especially, of the estimation or filtering error on optimal asset allocations. It is also
clear that commodities markets are highly volatile and that spot assets exhibit jumps, Hilliard and Reis
(1998). The impact of jumps in the optimal asset allocation with commodities remains an open
question. Finally a natural extension of this paper is to allow the investor to be constrained in the
commodity market, a main feature of the Traditional Hedging Model, Adler and Detemple (1988).
Appendix A. Cholevsky transformation

The four dimension Cholevsky transformation of our model is the following:

\[
\begin{bmatrix}
 dz_S \\ dz_f \\ dz_\delta \\ dz_\gamma
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 \rho_{sf} & \rho_{uf} & 0 & 0 \\
 \rho_{S\delta} & \rho_{u\delta} & \rho_{v\delta} & 0 \\
 0 & 0 & \rho_{v\gamma} & \rho_{v_\gamma}
\end{bmatrix}
\begin{bmatrix}
 dz_S \\ dz_f \\ dz_\delta \\ dz_\gamma
\end{bmatrix} = \sigma dz
\]

\[
\rho_{uf} = \sqrt{1 - \rho_{sf}^2} \quad \rho_{u\delta} = \frac{\rho_{fs} - \rho_{sf} \rho_{S\delta}}{\sqrt{1 - \rho_{sf}^2}} \quad \rho_{v\delta} = \frac{1 - \rho_{sf}^2 - \rho_{S\delta}^2 - \rho_{fs}^2 + 2 \rho_{sf} \rho_{S\delta} \rho_{fs}}{1 - \rho_{sf}^2}
\]

\[
\rho_{v\gamma} = \frac{\rho_{v\gamma}}{\rho_{v\delta}} \quad \rho_{v_\gamma} = \sqrt{1 - \left(\frac{\rho_{v\gamma}}{\rho_{v\delta}}\right)^2}
\]

The relation between the projection of the market of risk in the correlated and the orthogonal basis is the following:

\[
\lambda^{\text{pricing}} = \sigma^{-1} \lambda^{\text{pricing}} \quad \text{where} \quad \lambda^{\text{pricing}} = \lambda_0^{\text{pricing}} + \lambda_{v}^{\text{pricing}} Y \quad \text{is the natural market price of risk in the correlated basis.}
\]

\[
\lambda_0^{\text{pricing}} = \begin{bmatrix}
 \lambda_{S0} \\
 \lambda_{f0} \\
 \lambda_{c0} \\
 \lambda_{y0}
\end{bmatrix} \quad \text{and} \quad \lambda_{v}^{\text{pricing}} = \begin{bmatrix}
 0 & \lambda_{S\delta} & \lambda_{S\gamma} \\
 \lambda_{f\delta} & 0 & 0 \\
 0 & \lambda_{c\delta} & 0 \\
 0 & \lambda_{y\delta} & 0
\end{bmatrix}
\]

Remember that the set of invertible lower triangular matrix has a group structure so that \(\sigma^{-1}\) is also lower triangular. It follows that \(\sigma^{-1}\) is also lower triangular and \(\lambda_v\) and \(\lambda_c\) are functions of \(\lambda_S\), \(\lambda_f\), \(\lambda_c\) and then depend on the three state variables. Consequently \(\pi^{mv}_H\), which is a function of \(\lambda_c\) and \(\lambda_e\) - see appendix B, is a function of the three state variables.
Appendix B. Proofs of propositions

Because of the length of the computation, we only give a sketch of the proof.

As mentioned, the optimal wealth writes:

\[
W(t)^* = \int \frac{1}{\gamma} h^\gamma(t) \frac{1}{\gamma} \left( \frac{h^\gamma(t) \left( \frac{h^\gamma(T_i)}{h^\gamma(t)} \right)}{h^\gamma(T_i)} \right)^{\gamma-1} E_t \left[ \begin{array}{c} \frac{\lambda - \lambda^-}{\mu - \mu^-} \\ \frac{\mu - \mu^-}{\mu - \mu^-} \end{array} \right]
\]

Simple algebraic computation shows that:

\[
\lambda_{\min}^{\max} = \psi \lambda_{\text{pricing}}^{max}, \quad \psi = \frac{1}{\sigma_\delta^2 + \sigma_\delta^2} \begin{bmatrix}
\sigma_\delta^2 + \sigma_\delta^2 & 0 & 0 & 0 \\
0 & \sigma_\delta^2 + \sigma_\delta^2 & 0 & 0 \\
0 & 0 & \sigma_\delta^2 & \sigma_\delta^2 \sigma_\delta^2 \\
0 & 0 & \sigma_\delta^2 \sigma_\delta^2 & \sigma_\delta^2
\end{bmatrix},
\]

which sheds light on the importance of the idiosyncratic risk of the convenience yield – where we noted:

\[
\overline{\sigma_\delta} = \begin{bmatrix}
\sigma_\delta & \sigma_\delta & \sigma_\delta & \sigma_\delta
\end{bmatrix}
\]

As mentioned in the text, the wealth equals:

\[
W(t)^* = \int \frac{1}{\gamma} h(t) \frac{1}{\gamma} B(t; T_i) \gamma \frac{1}{\gamma} E_t \left[ \begin{array}{c} \exp \left[ \frac{1 - \gamma}{2 \gamma^2} \int_T T_i \| \hat{\lambda}_i - \sigma_i (\theta; T_i) \|^2 d\theta \right] \right] \right) \right)
\]

\[
E_t \left( \exp \left[ \frac{1 - \gamma}{2 \gamma^2} \int_T T_i \| \hat{\lambda}_i - \sigma_i (\theta; T_i) \|^2 d\theta \right] \right) = \exp \left( k(Y, T_i) \right)
\]

After differentiation:

\[
\frac{dW(t)^*}{W(t)} = \left[ \frac{1}{\gamma} \Lambda_i + \left( 1 - \frac{1}{\gamma} \right) \sigma_\delta(t; T_i) \gamma + \sigma_\gamma k_i(t, Y) \right] dz
\]

Identifying with the wealth constraint we get:
\[ \sigma' \pi = \gamma \Lambda + \left[ 1 - \frac{1}{\gamma} \right] \sigma_\beta(t; T_t) + \sigma_\gamma k(t, Y) \]

We project the above equality on the kernel of \( \sigma' \) and get that \( \hat{\lambda} = 0 \) and the optimal proportions:

\[ \pi = \frac{1}{\gamma} (\sigma \sigma')^{-1} \sigma \lambda(t, Y) + \left[ 1 - \frac{1}{\gamma} \right] (\sigma \sigma')^{-1} \sigma \sigma_\beta(t; T_t) + (\sigma \sigma')^{-1} \sigma \sigma_\gamma k(t, Y) \]

which proves proposition 1. Proposition 2 and 3 follow from simple algebraic computation.

We apply the Feynman-Cac Theorem to \( E_t \exp \left\{ \int_0^{T_t} [\lambda - \sigma_\beta(\theta; T_t)]^2 d\theta \right\} \) to obtain the following partial differential equation:

\[ k_{\beta t} = \frac{1 - \gamma}{2\gamma^2} \left| \lambda - \sigma_\beta(t) \right|^2 + k_{\gamma t} + \frac{1}{2} \left| \frac{1}{\gamma} \sigma_\gamma \gamma \right|^2 k_{\gamma t} + \frac{1}{2} tr \left[K_{\gamma \gamma} \sigma_\gamma \gamma \right], \quad k(0, Y) = 0, \]

where \( \mu_{\gamma}^{\beta} \) is derived from the dynamic of the state variables under \( \tilde{P} \) that, because of the affine structure of the market, writes:

\[ dY = \left[ \mu_{\gamma 0}(t) + \mu_{\gamma Y}(t) \right] dt + \sigma_Y dz \]

\[ \mu_{\gamma 0}(t) = \begin{bmatrix} \alpha \theta \\ \kappa \delta \\ \lambda_{SS} - \frac{\sigma_S^2}{2} \end{bmatrix}, \quad \mu_{Y\gamma}(t) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\kappa & 0 \\ 1 & \lambda_{SS} - 1 & \lambda_{SX} \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & \sigma_r & 0 & 0 \\ 0 & 0 & \sigma_c & \sigma_y \\ \sigma_s & 0 & 0 & 0 \end{bmatrix} \]

So that \( dY = \mu^{\beta} dt + \sigma_Y dz \) and \( \mu_t^{\beta} = \mu_{\gamma 0} + \mu_{\gamma Y} Y \) such that:
\[ \mu_{v0} = \mu_{0}(t) + \left( \frac{1}{\gamma} - 1 \right) \sigma_{Y} \left[ \sigma_{\alpha}(t) \right]_{\min/\max} \] and \[ \mu_{vY} = \mu_{Y} + \sigma_{Y} \left( \frac{1}{\gamma} - 1 \right) \lambda_{Y} \] where we can conveniently write \[ \lambda_{\min/\max} = \lambda_{Y}^0(t) + \lambda_{Y}^0(t)Y \] because of the Cholevsky transformation:

\[ \lambda_{0}^{\min/\max}(t) = \psi \sigma^{-1} \lambda_{\text{pricing}}^0, \quad \lambda_{Y}^{\min/\max}(t) = \psi \sigma^{-1} \lambda_{\text{pricing}}^Y. \]

We show by identification that \[ k(\tau, Y) = \alpha(\tau) + \beta(\tau)Y(t) + \frac{1}{2} Y(t)\gamma(\tau)Y(t) \] is a solution, where we do not need to specify \( \alpha(\tau) \) and where \( \beta(\tau) = \left[ \beta_\beta(\tau) \right] \) and \( \beta_x(\tau) \) are respectively vector and symmetric matrix that solves the following matrix ordinary differential system:

\[
\frac{d\gamma(\tau)}{d\tau} = 1 - \frac{\gamma}{\gamma^2} \lambda_\gamma(t)\lambda_\tau(t) + \gamma(\tau)\mu_\gamma + \mu_\gamma\gamma(\tau) + \gamma(\tau)\sigma_Y \sigma_Y \gamma(\tau) \quad \gamma(0) = 0, \]

\[
\frac{d\beta(\tau)}{d\tau} = 1 - \frac{\gamma}{\gamma^2} \lambda_\gamma(t)\left[ \sigma_\alpha(\tau) - \lambda_\alpha(t) \right] + \gamma(\tau)\mu_\alpha + \left[ \mu_\alpha + \gamma(\tau)\sigma_Y \right] \beta(\tau) \quad \beta(0) = 0.
\]

References


Table 1: numerical parameters

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Figure 1: $\pi_{S}^{MV} = function(T_t)$
Figure 2: $\pi_{H}^{MV} = function(T_t)$
Figure 3: $\pi_{B}^{MV} = function(T_t)$
Figure 4: $\pi_{B}^{Hedge\_rate} = function(T_t)$
Figure 5: $\pi_H^{\text{Hedge-MPR}} = \text{function}(T_i)$

![Figure 5](image)

Figure 6: $\pi_H^{\text{Hedge-MPR}} = \text{function}(T_i)$

![Figure 6](image)

Figure 7: $\pi_B^{\text{Hedge-MPR}} = \text{function}(T_i)$

![Figure 7](image)

Figure 8: $\pi_S^{\text{mv}} = \text{function}(\delta)$

![Figure 8](image)

Figure 9: $\pi_H^{\text{mv}} = \text{function}(\delta)$

![Figure 9](image)

Figure 10: $\pi_B^{\text{mv}} = \text{function}(\delta)$

![Figure 10](image)

Figure 11: $\pi_S^{\text{Hedge-MPR}} = \text{function}(T_i)$

![Figure 11](image)
Figure 12 : $\pi^H_{Hedge\_MPR} = function(\delta)$

Figure 13 : $\pi^B_{Hedge\_MPR} = function(\delta)$

Figure 14 : $\pi^B_{MV} = function(r)$

Figure 15 : $\pi^S_{MV} = function(X)$

Figure 16 : $\pi^H_{MV} = function(r)$

Figure 17 : $\pi^B_{MV} = function(X)$