

Impact of Stochastic Interest Rates and Stochastic Volatility on Variable Annuities

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With the success of variable annuities, insurance companies are piling up large risks in terms of both equity and fixed income assets. These risks should be properly modeled as the resulting dynamic hedging strategy is very sensitive to the modeling assumptions. The current literature has been largely focusing on simple variations around Black-Scholes model with basic interest rates term structure models. However, in a more realistic world, one should account for both Stochastic Volatility and Stochastic Interest rates. In this paper, we examine the combine effect of a Heston-type model for the underlying asset with a HJM affine stochastic interest rates model and apply it to the pricing of GMxB (GMIB, GMDB, GMAB and GMWB). We see that stochastic volatility and stochastic interest rates have an impact on the resulting fair value of the contract and the resulting fair fee as well as mainly on the vega hedge. Interestingly, using a stochastic volatility model leads to scenarios with high level of volatility for long maturities resulting in a higher contract value and a resulting fair fee. We also see that the impact of stochastic interest rates and volatility is more pronounced on the vega hedge than on the delta hedge.

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1 Introduction

In the recent years, variable annuities have become very popular life insurance products. Due to the commercial success of these products, exposure of life insurance companies to financial market risk has grown dramatically. As a result, they are now facing hedging issues similar to those of investment banks. Besides, literature on variable annuities has grown considerably in recent years. However, account modelisation framework has remained quite poor compared to what can be found in equity derivatives market. Yet, as empirical evidences by Bakshi et al [BC00] suggest, a careful modelisation of the asset evolution is of much importance when it comes to managing long-maturity products.

In this paper, we propose a stochastic volatility and stochastic interest rates model aimed at pricing and hedging variable annuities. Most of current life insurance contracts do not involve exotic interest rates derivatives: fixed-income risk originates from bonds and equities. Therefore, a Hull-White model may be sufficient to capture interest rates risk. Regarding equity risk, exposure is larger: policyholders have indeed the right to exit the contract at several times and strikes. Therefore, it is advisable to consider a model which is able to account for the smile and its dynamics. Stochastic volatility models, like Heston, SABR and derived version of these, are standard market choices when facing such issues. In this paper, we have chosen to combine a Hull-White model with a Heston model [Hes93]. Van Haastrecht et al. [VHS08] have conducted a similar work when studying the Schoebel-Zhu-Hull-White model [SZ99] and applying it to both Equity and FX derivatives. Besides having stochastic volatility and stochastic interest rates, this framework takes explicitly into account correlation between interest rates and asset diffusion.

This paper is structured as follows. First, we review variable annuities and present their major risks. In section 3, we take advantage of the great tractability of stochastic volatility model to exhibit a closed form GMAB pricing formula. Then, in section 4.2, we derive some closed-form formulas for specific variable annuities. Afterwards, we extend the Heston-Hull-White model to time-dependent parameters while preserving model tractability. Finally, in section 5, we perform an extensive empirical study of our model.

2 Variable Annuities

Fixed annuity used to be standard for life insurance contract. The policyholder received some predefined fixed amount, determined by his age and the interest rates level. In the 1970s, interest rates were still quite high. Therefore fixed annuities used to provide a good return to policyholders. There was no particular reason to go for risky assets. However, in the past two decades, as interest rates declined, fixed annuities became less appealing. Insurance companies decided to turn to the equity market to keep offering attractive return rates, giving birth to variable annuities.

Variable annuities cashflows are dependent of risky financial underlyings. Generally, the financial portfolio is composed of bonds, equity and funds. Even though these products may provide higher returns than fixed annuities, they can also lead to bad surprises. Over the last decade, a specific class of Variable Annuities, namely the Guaranteed Minimum Benefits, has become increasingly popular. Basically, these products are designed to guarantee a minimum performance level of the underlying, thus protecting the policyholder against market downfalls.

Pricing Guaranteed Minimum Benefits is quite challenging as it involves three major risks:

- mortality risk : life insurance payoffs are conditioned by survival probabilities. Therefore, life insurance resellers would like to hedge out their exposure to mortality risk. Currently, they are some attempts to launch a market for mortality bonds.
- policyholder risk : life insurance contracts may feature lapse and withdrawal options. A lapse option authorizes the policyholder to terminate the contract in advance (equivalent of an American option). A withdrawal option gives the policyholder the right to withdraw some money before the contract maturity. Pricing accurately these options involves a thorough statistical analysis of the policyholder behavior.
- financial risk : finally, variable annuities are indexed onto market instruments. As a result, there is a strong financial risk. Furthermore, they are often long-maturity derivatives (20 to 30 years in general), which makes modeling all the more complicated.

In this paper, we will study only the latter risk. When it comes to variable annuities pricing, the mortality risk will be taken into account through simple deterministic models. Readers who are also interested in mortality risk modeling are invited to read papers by Cairns et al [Dow08], Gouriéroux and Monfort [GM08] and Renshaw and Haberman [RH03]. These papers feature latest research insurance linked securities design and advanced mortality stochastic modeling.

Regarding policyholder risk, litterature is not very abundant. Life insurance companies have a statistical approach of the problem: they collect historical data on their previous contracts in order to predict future behaviours. Whereas financial risk and mortality risk are unlikely to be correlated, there may exist a strong correlation between policyholder risk and financial risk. Policyholder behaviors may indeed be strongly affected by market situation, especially in crisis time. As a result, pricing of such derivatives has to be scenario-based.

2.1 Guaranteed Minimum Benefits

2.1.1 Death Benefit

A death benefit (denoted by DB) corresponds to a cashflow G_t which value depends on a time t that is paid at the death of the policyholder. Usually, the cashflow equals the guaranteed amount at the death event. Most, but not all, Guaranteed Minimum Benefits embed a death benefit. In this case, we have:

$$DB = P(0, \tau) G_\tau 1_{\tau < T}.$$

where $P(0, t)$ denotes the risk free zero coupon bond price seen from 0 and paying 1 at time t , τ is the stochastic time corresponding to the death of the policyholder and T is the maturity of the contract.

2.1.2 Life Benefit

A life benefit refers to any cashflows that are paid provided that the policyholder is still alive at the end of the contract. The guaranteed amount may either be perceived as a single amount, called Amount Benefit (denoted by AB or GMAB), or as a life annuity, called Income Benefit (denoted by IB). At time T , the policyholder has the choice to receive either the guaranteed amount G_{IB} as a life income with conversion rate a^G or the account value A_T as a lump payment. The life annuity conversion rate a_G is set at time T according to both mortality risk and market conditions. We have the following relationships:

$$\begin{aligned} AB &= P(0, T) G_T 1_{\tau \geq T}, \\ IB &= P(0, T) \max \{ a^G G_T, A_T \} 1_{\tau \geq T}. \end{aligned}$$

In order to reflect the preference of the policyholder for liquidity, we could add a multiplicative factor. $a^G G_T$ must indeed be significantly greater than A_T for the policyholder to choose the life annuity.

The standard guarantee offered by life insurance companies is the return of premium meaning that the policyholder should at least receive the invested notional at maturity, ie $G_T = \max \{ A_T, A_0 \}$. In this case, we have :

$$\begin{aligned} AB &= P(0, T) (A_T + \max \{ A_0 - A_T, 0 \}) 1_{\tau \geq T}, \\ IB &= P(0, T) \max \left\{ 1, \frac{a^G}{a^M} \right\} (A_T + \max \{ A_0 - A_T, 0 \}) 1_{\tau \geq T}. \end{aligned}$$

We can remark that the Amount Benefit is nothing but a standard European put option on the account asset A_T with strike A_0 . Regarding the Income Benefit, it would also be a standard European put if there was not the a^M term. The market life annuity rate depends indeed of the interest rates and, therefore, is not independent of A_T in our modeling framework.

However, G_T may have a more complicated expression, if the policyholder wants to pay a higher annual fee.

2.1.3 Contract Options

Insurance companies propose two standard mechanisms to provide higher returns: the ratchet and the roll-up options. The roll-up option guarantees that the invested amount increases at a minimum fixed rate (typically, 5% or 6%) over the contract life:

$$G_T = \max \{(1 + R)^T A_0, A_T\}$$

It is important to notice that an Amount Benefit with roll-up option is still a vanilla European put option. Both the guaranteed payment and the strike are now equal to $(1 + R)^T A_0$.

The ratchet option guarantees that at least the maximum of account values at anniversary dates of the contract is received:

$$G_T = \max_{i=0,1,\dots,n} \{A_{t_i}\}.$$

3 Heston-Hull-White model

Hull-White model has been introduced in 1993 by Hull and White [HW93]. It assumes a mean-reverting diffusion of the short rate. Despite having some major disadvantages such as allowing negative short rates, it is still widely used for its great tractability. Heston [Hes93] published his stochastic volatility model in the very same year [Hes93]. The asset has a lognormal diffusion while the variance follows Cox-Ingersoll-Ross dynamics. It has become a standard market model to price equity derivatives with smile. Note that time-dependent variants may provide better fits than the original formulation. Combining these models leads to a three-factor hybrid model which variables are: the short rate $r(t)$, the asset price $S(t)$ and the asset volatility $v(t)$. The dynamics of this model writes:

$$dS(t) = r(t)S(t)dt + \sqrt{v(t)}S(t)dW_S(t) \quad (1)$$

$$dv(t) = \kappa[\psi - v(t)]dt + \epsilon\sqrt{v(t)}dW_v(t) \quad (2)$$

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW_r(t) \quad (3)$$

where the Brownian motions are correlated as follows:

$$\langle dW_S, dW_v \rangle = \rho_{Sv}$$

$$\langle dW_S, dW_r \rangle = \rho_{Sr}$$

$$\langle dW_v, dW_r \rangle = \rho_{vr}.$$

In the following sections, we are going to consider general payoffs which are a function of the asset price at maturity T . Hence, it may be preferable to express the dynamics of our model under the forward measure Q^T [GER96]. Then, we will find an analytical expression of the characteristic function of the asset price. Following the works of Carr & Madan [CM99] and Lewis [Lew00], we will be able to write the variable annuities payoffs in terms of the characteristic function ϕ .

3.1 Moving forward

In the Hull-White model, we have an analytical expression for the discount bond price:

$$P(t, T) = \exp[A_r(t, T) - B_r(t, T)r(t)]$$

where

$$A_r(t, T) = \ln\left(\frac{P(0, T)}{P(0, t)}\right) + B_r(t, T)f(0, t) - \frac{\sigma^2}{4a}(1 - e^{-2at}B_r(t, T)^2)$$

and

$$B_r(t, T) = \frac{1 - e^{-a(T-t)}}{a}.$$

Under the risk-neutral measure Q , the discount price follows the process

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma B_r(t, T)dW_r(t),$$

and the forward asset price is equal to:

$$F(t) = \frac{S(t)}{P(t, T)} = \frac{S(t)}{\exp[A_r(t, T) - B_r(t, T)r(t)]}.$$

Applying Ito's lemma to this equation, we get the dynamics of the asset price under the forward measure Q^T :

$$\begin{aligned} dF(t) &= (\sigma^2 B_r^2(t, T) + \rho_{Sr}\sqrt{V(t)}\sigma B_r(t, T))F(t)dt \\ &\quad + \sqrt{v(t)}F(t)dW_S(t) + \sigma B_r(t, T)F(t)dW_r(t). \end{aligned}$$

By definition, the forward asset price $F(t)$ is a martingale under the T -forward measure Q^T . This implies the following transformations on the Brownian motions:

$$\begin{aligned} dW_r(t) &\mapsto dW_r^T(t) - \sigma B_r(t, T)dt \\ dW_S(t) &\mapsto dW_S^T(t) - \rho_{Sr}\sigma B_r(t, T)dt \\ dW_v(t) &\mapsto dW_v^T(t) - \rho_{rv}\sigma B_r(t, T)dt \end{aligned}$$

Under Q^T , the processes for $F(t)$ and $V(t)$ are given by:

$$\begin{aligned} dF(t) &= \sqrt{v(t)}F(t)dW_S^T(t) + \sigma B_r(t, T)F(t)dW_r^T(t) \\ dV(t) &= \kappa(\xi - v(t))dt + \epsilon\sqrt{v(t)}dW_v^T(t) \end{aligned}$$

The diffusion equation of the forward asset price can be simplified by considering the logarithm of the forward asset price $x(t) = \ln F(t)$ rather than asset price $F(t)$ itself. This yields to the system :

$$dx(t) = -\frac{1}{2}v^x(t)dt + \sqrt{v(t)}dW_S^T(t) + \sigma B_r(t, T)dW_r^T(t) \quad (4)$$

$$dv(t) = \kappa(\xi - v(t))dt + \epsilon\sqrt{v(t)}dW_v^T(t) \quad (5)$$

with

$$v^x(t) = v(t) + 2\rho_{Sv}\sqrt{v(t)}\sigma B_r(t, T) + \sigma^2 B_r^2(t, T)$$

One great advantage of moving to the forward measure Q^T is that one variable has vanished. We are now left with a system of two variables $x(t)$ and $v(t)$. Let's now determine the characteristic function of this system.

3.2 Setting the problem

In order to find the characteristic function of the system described above, we apply the Feynman-Kac theorem. Basically, this enables to transform this problem into solving a partial differential equation.

The Feynman-Kac theorem tells that the characteristic function

$$\Phi(t, x, v) = \mathbb{E}^{Q^T} [\exp\{ivx(T)\} | \mathcal{F}_t]$$

is the solution of the following partial differential equation

$$0 = \Phi_t - \frac{1}{2}v^x(t)\Phi_x + \kappa[\xi - v(t)]\Phi_v + \frac{1}{2}v^x(t)\Phi_{xx} \quad (6)$$

$$+ \left(\rho_{Sv}\epsilon v(t) + \rho_{rv}\epsilon\sqrt{v(t)}\sigma B_r(t, T) \right) \Phi_{xv} + \frac{1}{2}\epsilon^2 v(t)\Phi_{vv}$$

$$\Phi(T, x, v) = \exp\{ivx(T)\}, \quad (7)$$

where the subscript denote partial derivatives. Note that this equation resembles much to the standard Heston partial differential equation, which writes:

$$0 = \Phi_t - \frac{1}{2}v(t)\Phi_x + \kappa[\xi - v(t)]\Phi_v + \frac{1}{2}v(t)\Phi_{xx} \quad (8)$$

$$+ \rho_{Sv}\epsilon v(t)\Phi_{xv} + \frac{1}{2}\epsilon^2 v(t)\Phi_{vv}$$

$$\Phi(T, x, v) = \exp\{ivx(T)\}. \quad (9)$$

Hence, we may guess that the solution must have a similar expression to that of Heston PDE.

4 Closed-form pricing of variable annuities

4.1 Transform-based pricing

4.1.1 Put option

Since Carr and Madan [CM99] have introduced Fourier transform based option pricing, extensive literature has been covering the topic. In this paper, we follow the work of Lewis [Lew00] and Lord & Kahl [LK08]. The price of a call option is given by

$$C(F, K, T) = F - \frac{e^{\eta k}}{\pi} \operatorname{Re} \int_0^\infty e^{-iuk} \frac{\Phi(u - i(\eta + 1))}{(u - i(\eta + 1))(u - i\eta)} du,$$

where F is the forward, K the strike, $k = \ln(K)$ and T is the option maturity¹. Ideally, the dampening parameter η could be chosen arbitrarily: it is introduced only to ensure that the integral is well-defined.

4.1.2 GMAB

As seen earlier, the GMAB (2.1.2) is merely a put option. The only difference lies in the dependence to mortality. Since there is no correlation between the account and the mortality rate in our setting, the payoff writes :

$$\text{GMAB}(T) = A(T) C(F, K, T), \quad (10)$$

where F is the forward account value, K the guaranteed amount, T the contract maturity and $A(T)$ the probability that the policyholder is alive at time T . The eq. (10) is well defined only if the guaranteed amount is known at $t = 0$. Therefore it is not valid when the contract embeds a ratchet option. In this specific case, the guaranteed amount depends indeed on the account value.

4.1.3 GMDB

In a GMDB, the benefit amount is paid at death of the policyholder. In a sense, it is similar to a Credit Default Swap product. Therefore, the price should theoretically be equal to

$$\text{GMDB}(T) = \int_0^T D(t) C(F, K, t) dt$$

where $D(t)$ is the instantaneous death rate. However, for practical considerations, the amount can only be paid at the discrete times $T_1, T_2, \dots, T_n = T$. A more pragmatic price writes :

$$\text{GMDB}(T) = \sum_{i=0}^{N-1} D(T_{i-1}, T_i) C(F, K, T_i) \quad (11)$$

where $D(T_{i-1}, T_i) = \int_{T_{i-1}}^{T_i} D(t) dt$ is the probability that the policyholder dies between T_{i-1} and T_i .

We have managed to establish of closed-form expressions for standard variable annuities, which are valid for any stochastic model provided that we know the characteristic function of the account value.

4.2 Characteristic function

4.2.1 No correlation

First, we will consider the no correlation case: this will enable us to find an exact price in the uncorrelated case. Ideally, any approximation which includes correlation should

¹the corresponding put option is immediately deduced by call put parity

be consistent with this formula in $\rho = 0$. Later on, we will build on this initial solution to construct our approximated closed-form solution.

When correlations ρ_{S_r} and ρ_{v_r} are equal to 0, the equation (6) is solvable. We can indeed write the diffusion equation of x (4) as:

$$dx(t) = -\frac{1}{2}v(t)dt + \sqrt{v(t)}dW_S^T(t) - \frac{1}{2}\sigma^2 B_r^2(t, T)dt + \sigma B_r(t, T)dW_r^T(t) \quad (12)$$

$$= dx_s + dx_r \quad (13)$$

where x_s and x_r are two independent stochastic processes. This separation into two variables makes the solving much easier.

Proposition 1. *Under the no-correlation assumption, the characteristic function of the forward log-asset price is given by :*

$$\begin{aligned} \Phi(t, x, u) &= \mathbb{E}^{Q^T} \left[e^{iux(T)} | \mathcal{F}_t \right] \\ &= \mathbb{E}^{Q^T} \left[e^{iux_s(T)} | \mathcal{F}_t \right] \times \mathbb{E}^{Q^T} \left[e^{iux_r(T)} | \mathcal{F}_t \right] \\ &= \Phi_s(t) \times \Phi_r(t) \end{aligned}$$

where x_s is a Heston process and x_r a normal process with time-dependent drift and volatility and refer to the standard articles on the subject [Hes93], [Lew00], [KJ05], [MN03].

Both processes have been largely reviewed in the financial literature. Therefore, we will not detail too much the calculations.

Proposition 2. *The characteristic function of the normal process x_r is equal to*

$$\Phi_r(t) = E^{Q^T} \left[e^{iux_r(t)} | \mathcal{F}_t \right] = e^{-\frac{1}{2}u(i+u)V(t, T)}$$

with

$$V(t, T) = \frac{\sigma^2}{a^2} \left((T-t) + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a} \right).$$

Proof. The characteristic function of a normal process of drift $\mu(t)$ and volatility $\sigma(t)$ is equal to:

$$\Phi(t) = e^{iu \int_t^T \mu(s)ds - \frac{u^2}{2} \int_t^T \sigma^2(s)ds}.$$

Thus, the characteristic function of x_r writes:

$$\begin{aligned} \Phi_r(t) &= e^{-\frac{1}{2}u(i+u) \int_t^T \sigma^2 B_r^2(s, T)dt} \\ &= e^{-\frac{1}{2}u(i+u)V(t, T)} \end{aligned}$$

with

$$\begin{aligned} V(t, T) &= \frac{\sigma^2}{a^2} \int_t^T \left[1 - e^{a(T-s)} \right]^2 dt \\ &= \frac{\sigma^2}{a^2} \left((T-t) + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a} \right). \end{aligned}$$

□

Proposition 3. *The characteristic function of x_s is equal to:*

$$\Phi_s(t) = \mathbb{E} \left[e^{iux_s(T)} | \mathcal{F}_t \right] = \exp\{A(u, \tau) + B(u, \tau)v(t) + iux(t)\}$$

where

$$\begin{aligned} A(u, \tau) &= \frac{\kappa\xi}{\epsilon^2} \left((\beta - d)\tau - 2 \ln \frac{ge^{-d\tau} - 1}{g - 1} \right) \\ B(u, \tau) &= \frac{(\beta - d)(1 - e^{-d\tau})}{(\beta + d)(1 - ge^{-d\tau})} \end{aligned}$$

with

$$\begin{aligned} \alpha &= -\frac{1}{2}(iu - u^2) & \gamma &= \frac{1}{2}\epsilon^2 \\ \beta &= \kappa - i\rho\epsilon u & d &= \sqrt{\beta - 4\alpha\gamma}. \end{aligned}$$

Proof. If we plug our solution into Heston PDE, we find that the functions A and B satisfy the following system of ODEs:

$$\begin{aligned} \frac{dA}{d\tau} &= \kappa\xi B \\ \frac{dB}{d\tau} &= \alpha - \beta B + \gamma B^2 \\ A(u, 0) &= 0 \\ B(u, 0) &= 0 \end{aligned}$$

with:

$$\begin{aligned} \alpha &= -\frac{1}{2}(iu - u^2) \\ \beta &= \kappa - i\rho\epsilon u \\ \gamma &= \frac{1}{2}\epsilon^2 \end{aligned}$$

One can notice that the second equation is of Ricatti type:

$$\frac{dB}{d\tau} = \gamma(B - a)(B - b)$$

where we have introduced the roots of the Riccati equation $a = (\beta + d)/\epsilon^2$, $b = (\beta - d)/\epsilon^2$ and the discriminant $d = \sqrt{\beta - 4\alpha\gamma}$. Therefore, we can immediately write the solution of this equation:

$$\begin{aligned} B(u, \tau) &= ab \frac{1 - e^{(b-a)\gamma\tau}}{a - be^{(b-a)\gamma\tau}} \\ &= \frac{\beta - d}{\beta + d} \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}. \end{aligned}$$

By integration, we obtain the solution to the first equation:

$$A(u, \tau) = \frac{\kappa \xi}{\epsilon^2} \left((\beta - d)\tau - 2 \ln \frac{1 - g e^{-d\tau}}{1 - g} \right)$$

When reading other articles, you may find different ways of writing the solution to Heston's PDE. You must be aware of complex discontinuities due to some of these formulations, as explained in Lord and Kahl [LK08]. Their solution, which is presented here, is not affected by such discontinuities. \square

4.2.2 Making the PDE simpler

Despite having found an exact formula for the characteristic function in previous paragraph, we may continue our investigations. We have indeed assumed that the correlation between interest rates and account value was zero. However, in most practical situations, the correlation has no reason to be close to zero. Therefore, there is definitely a need to derive an approximation in the non-zero correlation case.

First let recall the PDE verified by the characteristic function :

$$\begin{aligned} 0 &= \Phi_t - \frac{1}{2} [v(t) + 2\rho_{Sv}f_v(t) + \sigma^2 B_r^2(t, T)] \Phi_x \\ &\quad + \kappa[\xi - v(t)]\Phi_v + \frac{1}{2} [v(t) + 2\rho_{Sv}f_v(t) + \sigma^2 B_r^2(t, T)] \Phi_{xx} \\ &\quad + (\rho_{Sv}\epsilon v(t) + \rho_{rv}\epsilon f_v(t)) \Phi_{xv} + \frac{1}{2}\epsilon^2 v(t)\Phi_{vv} \\ \Phi(T, x, u) &= \exp(iux(T)) \end{aligned}$$

with

$$f_v(t) = \sqrt{v(t)}\sigma(t)B_r(t, T).$$

In this equation, the annoying fact is clearly the dependence in $\sqrt{v(t)}$. In order to bypass this problem, we propose to replace $f_v(t)$ by a deterministic function $f(t)$. In the following sections, we will study two choices for this function $f(t)$. The intuition is that this perturbative term should not affect too much the pricing and can be replaced by a deterministic function.

Proposition 4. *Let $f(t)$ be a deterministic time-dependent function. The solution of the following PDE:*

$$\begin{aligned} 0 &= \Phi_t - \frac{1}{2} [v(t) + 2\rho_{Sv}f(t) + \sigma^2 B_r^2(t, T)] \Phi_x \\ &\quad + \kappa[\xi - v(t)]\Phi_v + \frac{1}{2} [v(t) + 2\rho_{Sv}f(t) + \sigma^2 B_r^2(t, T)] \Phi_{xx} \\ &\quad + (\rho_{Sv}\epsilon v(t) + \rho_{rv}\epsilon f(t)) \Phi_{xv} + \frac{1}{2}\epsilon^2 v(t)\Phi_{vv} \end{aligned}$$

is given by the following closed-form solution:

$$\Phi(t, x, v) = \exp \{A(u, \tau) + B(u, \tau)v(t) + iux(t)\} \quad (14)$$

where

$$\begin{aligned} A(u, t, T) &= \frac{\kappa\xi}{\epsilon^2} \left[(\beta - d)\tau - 2 \ln \frac{1 - ge^{d\tau}}{1 - g} \right] - \frac{1}{2}u(i + u)V(t, T) - I(f) \\ B(u, t, T) &= \frac{\beta - d}{\beta + d} \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}} \end{aligned}$$

with

$$\begin{aligned} \alpha &= -\frac{1}{2}(iu - u^2) \\ \beta &= \kappa - i\rho\epsilon u \\ \gamma &= \frac{1}{2}\epsilon^2 \\ d &= \sqrt{\beta^2 - 4\alpha\gamma} \\ \tau &= T - t \\ I(f) &= \int_t^T f(s) (\rho_{Sv}u(i + u) + \rho_{rv}\epsilon iuB(u, s)) ds. \end{aligned}$$

Proof. The partial derivatives of our solution in eq. (14) are given by:

$$\begin{aligned} \Phi_t &= (A_\tau + B_\tau v)\Phi & \Phi_{xx} &= -u^2\Phi \\ \Phi_x &= iu\Phi & \Phi_{xv} &= iuB\Phi \\ \Phi_v &= B\Phi & \Phi_{vv} &= B^2\Phi \end{aligned}$$

Once these derivatives are plugged into the PDE, we obtain the system:

$$\begin{aligned} 0 &= (A_\tau + B_\tau v) + \frac{1}{2} [v(t) + 2\rho_{Sv}f(t) + \sigma^2 B_r^2(t, T)] iu \\ &+ \kappa[\xi - v(t)]B - \frac{1}{2} [v(t) + 2\rho_{Sv}f(t) + \sigma^2 B_r^2(t, T)] u^2 \\ &+ (\rho_{Sv}\epsilon v(t) + \rho_{rv}\epsilon f(t)) iuB + \frac{1}{2}\epsilon^2 v(t)B^2 \end{aligned}$$

Collecting the terms in $v(t)$ leads to a set of two ordinary differential equations:

$$\begin{aligned} 0 &= A_\tau + (\rho_{Sv}f(t) + \frac{1}{2}\sigma^2 B_r^2(t, T))iu + \xi B \\ &- (\rho_{Sv}f(t) + \frac{1}{2}\sigma^2 B_r^2(t, T))u^2 + \rho_{rv}\epsilon f(t)iuB \\ 0 &= B_\tau + \frac{1}{2}iu - \kappa B - \frac{1}{2}u^2 + \rho_{Sv}\epsilon iuB + \frac{1}{2}\epsilon^2 B^2 \end{aligned}$$

The second equation is a Riccati equation with constant coefficients:

$$\begin{aligned} B_t &= -\frac{1}{2}u(i-u) + (\kappa - \rho_{Sv}\epsilon iu)B - \frac{1}{2}\epsilon^2 B^2 \\ &= \gamma(B-a)(B-b) \end{aligned}$$

where

$$\begin{aligned} \gamma &= \frac{1}{2}\epsilon^2 \\ a &= \frac{\kappa - \rho_{Sv}\epsilon iu + \sqrt{d}}{\epsilon^2} \\ b &= \frac{\kappa - \rho_{Sv}\epsilon iu - \sqrt{d}}{\epsilon^2} \\ d &= (\kappa - \rho_{Sv}\epsilon iu)^2 - \epsilon^2 u(i-u). \end{aligned}$$

The solution of this equation writes :

$$B(u, t, T) = \frac{\beta - d}{\beta + d} \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}$$

with

$$\begin{aligned} \alpha &= -\frac{1}{2}(iu - u^2) \\ \beta &= \kappa - i\rho\epsilon u \\ \gamma &= \frac{1}{2}\epsilon^2 \\ d &= \sqrt{\beta^2 - 4\alpha\gamma} \end{aligned}$$

Finally, we are left with the first equation to solve. Despite being very simple, it is not very tractable:

$$\begin{aligned} A(u, t, T) &= \\ & - \int_t^T \left(\rho_{Sv} iu f + \frac{1}{2}\sigma^2 iu B_r^2(s, T) + \xi B - \rho_{Sv} f u^2 + \frac{1}{2}\sigma^2 B_r^2(s, T) u^2 + \rho_{rv} \epsilon f iu B \right) ds \\ &= - \int_t^T \left(\frac{1}{2}\sigma^2 B_r^2(s, T) u(i+u) + \xi B \right) ds - \int_t^T f(s) (\rho_{Sv} u(i+u) + \rho_{rv} \epsilon iu B) ds \\ &= \frac{\kappa\xi}{\epsilon^2} \left((\beta - d)\tau - 2 \ln \frac{1 - ge^{d\tau}}{1 - g} \right) - \frac{1}{2}u(i+u)V(t, T) - I(f) \end{aligned}$$

The terms have been reordered in order to highlight the three components of $A(u, t, T)$:

- the stochastic volatility term, equal to that found in Proposition 3
- the interest rates volatility term, identical to that found in Proposition 2

– a correlation term

□

The term that needs to be approximated is $I(f)$ which writes:

$$\begin{aligned} I(f) &= \int_t^T f(s) (\rho_{Sv}u(i+u) + \rho_{rv}\epsilon i u B) ds \\ &= \rho_{xv}u(i+u)I_1 + \rho_{rv}\epsilon i u I_2 \end{aligned}$$

where we have defined the two integrals: $I_1 = \int_t^T f(t)dt$ and $I_2 = \int_t^T B(t)f(t)dt$.

4.2.3 First deterministic approximation

In this first approach, we propose to replace the dependence in $\sqrt{v(t)}$ by $\sqrt{\mathbb{E}[v(t)]}$. Since the volatility follows a Cox-Ingersoll-Ross process, we know that

$$\mathbb{E}[v(t)] = v(0)e^{-\kappa t} + \xi(1 - e^{-\kappa t})$$

Despite being quite complicated, I_1 has an analytical expression :

$$I_1 = \frac{\sqrt{e^{-\kappa t}\xi}}{\kappa} \left(\kappa t \sqrt{\xi} - 2\sqrt{\mathbb{E}[v(t)]} + 2\sqrt{\xi} \ln \left[\xi + \sqrt{\xi} \sqrt{\mathbb{E}[v(t)]} \right] \right) \quad (15)$$

Unfortunately, this is not the case for I_2 . For numerical approximation, we will proceed to a Gauss-Legendre integration in order to reduce computation time.

In this first approximation, we face a major issue. Our approximation may not work well since we have postulated that $\sqrt{v(t)} \approx \sqrt{\mathbb{E}[v(t)]}$ although the square root is not a linear function.

4.2.4 Second deterministic approximation

In the first approximation, we have replaced the volatility term by the square root of the expected variance. However, we can imagine that replacing it by the expected volatility should lead to much better results and performances.

Applying Ito's lemma, we obtain the stochastic diffusion of the volatility process $\nu(t)$:

$$\begin{aligned} d\nu(t) &= \frac{1}{2}\kappa \left[\left(\xi - \frac{\epsilon^2}{\kappa} \right) \frac{1}{\nu(t)} - \nu(t) \right] dt + \frac{1}{2}\epsilon dW_v \\ &= \kappa_\nu [\xi_\nu - \nu(t)]dt + \epsilon_\nu dW_v. \end{aligned}$$

By moment-matching, we have that

$$\mathbb{E}[\nu(t)^2] = \mathbb{E}[v(t)] = Var[\nu(t)] + \mathbb{E}[\nu(t)]^2$$

with

$$\begin{aligned} \mathbb{E}[v(t)] &= M = \xi + (v(0) - \theta)e^{-\kappa t} \\ Var[\nu(t)] &= V = \frac{\epsilon^2}{4\kappa}(1 - e^{-\kappa t}) \end{aligned}$$

Heston	Initial vol	30%
	Long term vol	15%
	Mean reversion	30%
	Vol of vol	90%
	Correlation	-50%
Hull and White	Mean reversion	1%
Market data	Short rate	4%
	IR-equity correlation	-20%
	Contract maturity	15y

Table 1: Parameter values used in our numerical results

This technique has already been used by Zhu [Zhu08] in view of finding an efficient discretization scheme of Heston processes. Thus, we can deduce that

$$\begin{aligned}
\mathbb{E}[\nu(t)] &= \sqrt{M - V} \\
&= \sqrt{v(0)e^{-\kappa t} + \xi(1 - e^{-\kappa t}) - \frac{\epsilon^2}{4\kappa}(1 - e^{-\kappa t})} \\
&= \sqrt{v(0)e^{-\kappa t} + \left(\xi - \frac{\epsilon^2}{4\kappa}\right)(1 - e^{-\kappa t})}
\end{aligned}$$

As expected, this expression looks very similar to that of the first approximation. The only difference comes from the last term $\frac{\epsilon^2}{4\kappa}(1 - e^{-\kappa t})$. Thus, we can use the analytical expression found in the first deterministic approximation by replacing ξ with $\xi - \frac{\epsilon^2}{4\kappa}$ in eq. (15).

Grzelak and Oosterlee [GO09], who have also studied independently the Heston model with stochastic interest rates, have obtained a similar approximation for the expected volatility:

$$\mathbb{E}[\nu(t)] = \sqrt{v(0)e^{-\kappa t} + \xi(1 - e^{-\kappa t}) - \frac{\epsilon^2}{8\kappa} \frac{(1 - e^{-\kappa t})(\xi(e^{\kappa t} - 1) + 2\nu(0))}{\xi(e^{\kappa t} - 1) + \nu(0)}}.$$

5 Numerical results

In this section, we present numerical results. For all our numerical results, we use the parameters values provided in table 1. In particular, we assume an interest rate of 4%, a 1% mean reversion in Hull and White, a default correlation between interest rates and equity of -20% and a contract maturity of 15 years.

In tables 2, respectively 3 and 4, we benchmark our two approximated formulae against Monte-Carlo simulations on the put option for a maturity of 15 years, respectively 20

IR vol	Equity vol	Correl	Monte Carlo	Approx1 error	Approx2 error
0.3%	0.2	-0.2	13.033%	1.01%	0.11%
-	-	0	13.131%	1.09%	0.12%
-	-	0.2	13.229%	1.17%	0.13%
-	0.3	-0.2	13.794%	1.58%	0.26%
-	0.4	-0.2	14.843%	2.27%	0.50%
0.6%	0.2	-0.2	13.211%	1.16%	0.13%
-	0.3	-0.2	13.919%	1.67%	0.28%
-	0.4	-0.2	14.918%	2.32%	0.51%
0.9%	0.2	-0.2	13.569%	1.43%	0.16%
-	0.3	-0.2	14.222%	1.87%	0.31%
-	0.4	-0.2	15.164%	2.47%	0.55%
1.2%	0.2	-0.2	14.101%	1.79%	0.20%
-	0.3	-0.2	14.695%	2.17%	0.36%
-	0.4	-0.2	15.577%	2.73%	0.61%

Table 2: Comparison of our approximations with Monte Carlo for 15 years maturity

and 30 years. For the Monte Carlo simulation, we use 50,000 simulations with antithetic reduction variance to have a good precision. We can see that the first approximation (which results are given in the column entitled Approx1) error is not very good and can lead to relative errors of more than 2% while the second approximation performs better. This makes sense as the second approximation uses the second order moment to match the distribution while the first method relies only on the first moment. Overall, we can conclude that the second method performs well with a maximum relative error of 0.60%.

We then discuss in figure 1 the impact of stochastic interest rates for negative correlation (-20%) and for positive correlation (20%). We can see that the impact of stochastic interest rates is quite different for negative and positive interest rates. This can be easily explained as the put option is sensitive to the total volatility σ_{tot} which value can be intuitively given by

$$\sigma_{tot}^2 = \sigma_{eq}^2 + \sigma_{ir}^2 + 2\rho_{eq,ir}\sigma_{eq}\sigma_{ir} \quad (16)$$

where σ_{eq} is the equity volatility, σ_{ir} is the interest rates volatility, $\rho_{eq,ir}$ the correlation between interest rates and equity. From equation 16, we see that the total volatility is an increasing function of the interest rates volatility in case of positive correlation $\rho_{eq,ir}$ while it is decreasing and then increasing in the case of negative correlation. As for the correlation impact, we see that the total volatility σ_{tot} (and hence the put option) is an increasing function of the correlation as confirmed by table 2.

Last but not least, we present the fair fees computed as the annual fees to pay in the life insurance contract to cover the cost of the embedded put option. As expected, the fair fee decreases with the maturity of the contract and increases with the additional

IR vol	Equity vol	Correl	Monte Carlo	Approx1 error	Approx2 error
0.3%	20%	-20%	12.337%	0.958%	0.087%
-	-	0%	12.407%	1.034%	0.094%
-	-	20%	12.476%	1.106%	0.101%
-	30%	20%	13.081%	1.650%	0.226%
-	40%	20%	13.892%	2.283%	0.417%
0.6%	20%	20%	12.739%	1.356%	0.124%
-	30%	20%	13.372%	1.884%	0.258%
-	40%	20%	14.205%	2.517%	0.460%
0.9%	20%	20%	13.127%	1.687%	0.154%
-	30%	20%	13.782%	2.201%	0.301%
-	40%	20%	14.633%	2.834%	0.517%
1.2%	20%	20%	13.632%	2.086%	0.190%
-	30%	20%	14.307%	2.593%	0.355%
-	40%	20%	15.170%	3.230%	0.590%

Table 3: Comparison of our approximations with Monte Carlo for 20 years maturity

IR vol	Equity vol	Correl	Monte Carlo	Approx1 error	Approx2 error
0.3%	20%	-20%	10.045%	0.390%	0.024%
-	-	0%	10.083%	0.420%	0.026%
-	-	20%	10.120%	0.448%	0.028%
-	30%	-20%	10.339%	0.593%	0.055%
-	40%	-20%	10.749%	0.822%	0.101%
0.6%	20%	-20%	10.113%	0.443%	0.027%
-	30%	-20%	10.387%	0.623%	0.058%
-	40%	-20%	10.779%	0.837%	0.103%
0.9%	20%	-20%	10.252%	0.539%	0.033%
-	30%	-20%	10.505%	0.691%	0.064%
-	40%	-20%	10.876%	0.886%	0.109%
1.2%	20%	-20%	10.458%	0.664%	0.041%
-	30%	-20%	10.691%	0.791%	0.073%
-	40%	-20%	11.040%	0.968%	0.119%

Table 4: Comparison of our approximations with Monte Carlo for 30 years maturity

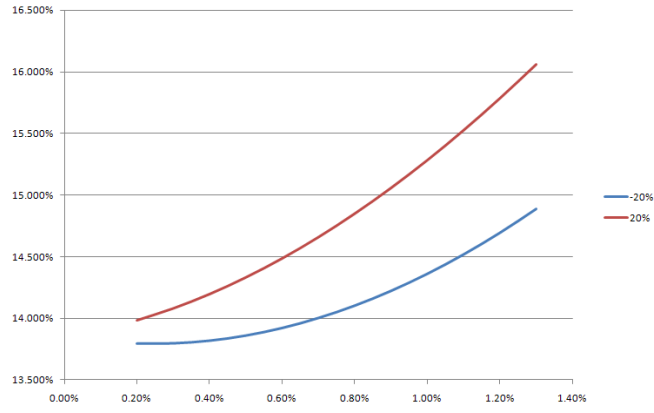


Figure 1: Impact of interest volatility for correlations of -20% and 20%

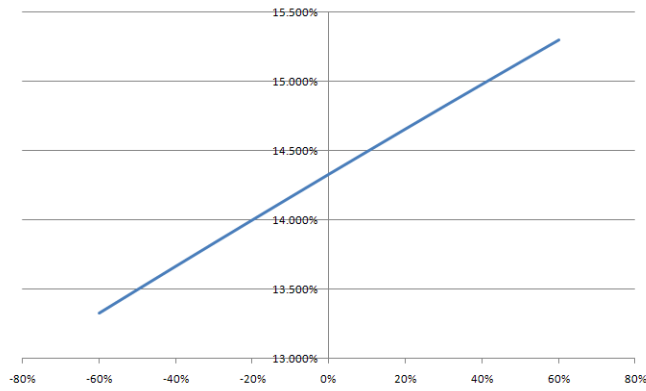


Figure 2: Impact of correlation between interest volatility and equity

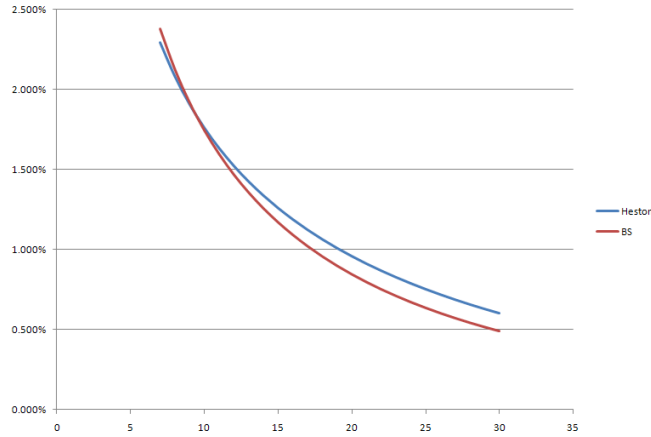


Figure 3: Fair fee for a Black Scholes and Heston Hull and White model for various maturities

Greeks	Stochastic	Deterministic	Difference
delta	-7.77%	-7.75%	0.31%
vega	8.54%	9.70%	-11.94%

Table 5: Greek comparison for stochastic and deterministic interest rates

volatility arising from stochastic interest rates. We compare our Heston Hull and White model with a simple Black Scholes model calibrated to match the volatility of Heston for the maturities 7 to 30 years. We find a Black Scholes volatility of 27.12%. As the Black Scholes model cannot reproduce the overall volatility of the Heston model, its implied volatilities are slightly higher on the short term and slightly lower on the long term. The resulting fair fees resulting are slightly higher on the short term for the Black Scholes model and slightly lower on long maturities. This suggests that using a non stochastic model will tend to underprice the value of the guarantee option on long maturities as shown in table 3. We can notice that the impact can be significant on 30 years contract as the difference of fee is about 10 basis point running or 22 percent in relative terms of the total cost. What is even more interesting is the impact of the modeling assumptions in terms of Greeks (delta and vega). In table 5 and 6, we see that the impact is very pronounced for the vega. The impact of stochastic interest rates in the Heston Hull and White model is very little. This makes sense as the put option is deeply out of the money.

Greeks	Heston	BS	Difference
delta	-7.77%	-13.80%	-43.69%
vega	8.54%	12.13%	-29.56%

Table 6: Greek comparison for Hull and White Heston model and Black Scholes model

6 Conclusion

In this paper, we have combined a Hull-White and a Heston model to account for both stochastic interest rates model and realistic shape of the equity smile. Under zero correlation, we can derive an explicit formula. When correlation is not null, we can find good approximation. The second order approximation provides good results when compared to Monte Carlo simulation. We finally analyze in the numerical section the impact of the stochastic interest rates and correlation between interest rates and equity.

As expected, we found that stochastic interest rates volatility can have a significant impact both in terms of the contract price and consequently the resulting fair fee to charge as well as the dynamic hedging strategy through a different delta and vega. We find that the impact of stochastic interest rates volatility is more pronounced for longer maturities and for the vega. We also see that the impact of stochastic volatility is substantial for long maturities and can lead to a different view of the risk profile as the fair fee is increased by 22%. These remarks are particularly important for Withdrawal Benefits, which have very long maturities. However, since standard models tend to overestimate vega risk, market participants wonder whether it is worth hedging it.

7 Future work

In this paper, we have made the standard hypothesis that the financial market is complete. Usually, this assumption is considered to be acceptable. However, regarding life insurance products, it may be inexact. Some underlying funds are indeed not very liquid: they can't be fully hedged out. Under incomplete market, the hedging strategy must be modified. This issue could be an interesting subject for future research.

Another interesting topic is the calibration of such a model. As a matter of fact, calibration of Heston model must take into account the stochastic behavior of interest rates. The closed-form formulas that have been derived in this paper may help much in finding an efficient calibration of this model.

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