Top-Down versus Bottom-Up Approaches in Risk Management

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Abstract:
Banks and other financial institutions face the necessity to merge the economic capital for credit risk, market risk, operational risk and other risk types to one overall economic capital number to assess their capital adequacy in relation to their risk profile. Beside just adding the economic capital numbers or assuming multivariate normality, the top-down and the bottom-up approach have been emerged recently as more sophisticated methods for solving this problem. In the top-down approach, copula functions are employed for linking the marginal distributions of profit and losses resulting from different risks. In contrast, in the bottom-up approach, different risk types are modeled and measured simultaneously in one common framework. Thus, there is no need for a later aggregation of risk-specific economic capital numbers.
In this paper, these two approaches are compared with respect to their ability to predict loss distributions correctly. We find that the top-down approach can underestimate the true risk measures for lower investment grade issuers. The accuracy of the marginal loss distributions, the employed copula function, and the loss definitions have an impact on the performance of the top-down approach. Unfortunately, given limited access to times series data of market and credit risk loss returns, it is rather difficult to decide which copula function an adequate modelling approach for reality is.

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1. Introduction

Banks are exposed to many different risk types due to their business activities, such as credit risk, market risk, or operational risk. The task of the risk management division is to measure all these risks and to determine the necessary amount of economic capital which is needed as a buffer to absorb unexpected losses associated with each of these risks. Predominantly, the necessary amount of economic capital is determined for each risk type separately. That is why the problem arises how to combine these various amounts of capital to one overall capital number.

Considering diversification effects requires to model the multivariate dependence between the various risk types. In practice, some kind of heuristics, based on strong assumptions, are often used to merge the economic capital numbers for the various risk types into one overall capital number.[2] For example, frequently, it is assumed that the loss distributions resulting from the different risk types are multivariate normally distributed. However, this is certainly not true for credit or operational losses. Two theoretically more sound approaches consist in the so-called top-down and bottom-up approaches. Both approaches are a step towards an enterprise-wide risk management framework, which can support management decisions on an enterprise-wide basis by integrating all relevant risk components.

Within the top-down approach, the separately determined marginal distributions of losses resulting from different risk types are linked by copula functions. The difficulty is to choose the correct copula function, especially given the limited access to adequate data. Furthermore, there are complex interactions between various risk types, for example between market and credit risk, in bank portfolios. Changes in market risk variables, such as risk-free interest rates, can have an influence on the default probabilities of obligors, or the development of the underlying market risk factor determines the exposure at default of OTC-derivatives with counterparty risk. It might suggest itself that copula functions are likely to only insufficiently represent this complex interaction because all

[2] For an overview on risk aggregation methods used in practice, see *Joint Forum* (2003), *Bank of Japan* (2005), and *Rosenberg* and *Schuermann* (2006).

interaction has to be pressed into some parameters of the (parametrically parsimoniously chosen) copula and the functional form of the copula itself.

In contrast, bottom-up approaches model the complex interactions described above already on the level of the individual instruments and risk factors which should make them more exact. These approaches allow to determine simultaneously, in one common framework, the necessary amount of economic capital needed for different risk types (typically credit and market risk), whereby possible stochastic dependencies between risk factors can directly be taken into account. Thus, there is no need for a later aggregation of the risk-specific loss distributions by copulas.

In this paper, we deal with the question how large the difference between economic capital computations based on top-down and bottom-up approaches is for the market and credit risk of banking book instruments. In order to focus on the differences caused by the different methodological approaches, we generate market and credit loss data with a simple example of a bottom-up approach for market and credit risk. Afterwards, the top-down approach is estimated and implemented based on the generated data and the resulting integrated loss distribution is compared with that one of the bottom-up approach. Thus, it is assumed that the bottom-up approach represents the real-world data generating process and we evaluate the performance of the top-down approach relative to the bottom-up approach. Doing this, we can ensure that the observed differences between the loss distributions are not overlaid by estimation and model risk for the bottom-up approach, but are only due to inaccuracies of the top-down approach.

The paper is structured as follows: in section 2, relevant literature with respect to the top-down and the bottom-up approach is reviewed. Afterwards, in sections 3 and 4, the model set up and the methodology of the comparison are explained. In section 5, the results are presented, and finally, in section 6, the main conclusions are summarized.

2. Literature Review

Sound approaches for risk aggregation can roughly be classified according to the two groups already mentioned in section 1. Let us start with briefly reviewing bottom-up approaches. Papers of this group exclusively deal with a combined treatment of the two risk types 'market risk' and 'credit risk'. *Kiesel* et al. (2003) analyze the consequences from adding rating-specific credit spread risk to the CreditMetrics model for a portfolio of defaultable zero coupon bonds. The rating transitions and the credit spreads are assumed to be independent. Furthermore, the risk-free interest rates are non-stochastic as in the original CreditMetrics model. However, *Kijima* and *Muromachi* (2000) integrate interest rate risk into an intensity-based credit portfolio model. *Grundke* (2005) presents a modified CreditMetrics model with correlated interest rate and credit spread risk. He also analyzes to which extent the influence of additionally integrated market risk factors depends on the model's parameterization and specification. *Jobst* and *Zenios* (2001) employ a similar model as *Kijima* and *Muromachi* (2000), but additionally introduce independent rating migrations. Beside the computation of the future distribution of the credit portfolio value, *Jobst* and *Zenios* (2001) study the intertemporal price sensitivity of coupon bonds to changes in interest rates, default probabilities and so on, and they deal with the tracking of corporate bond indices. This latter aspect is also the main focus of *Jobst* and *Zenios* (2003). Dynamic asset and liability management modelling under credit risk is studied by *Jobst* et al. (2006). *Barth* (2000) computes by Monte Carlo simulations various worst-case risk measures for a portfolio of interest rate swaps with counterparty risk. *Arvanitis* et al. (1998) and *Rosen* and *Sidelnikova* (2002) also account for stochastic exposures when computing the economic capital of a swap portfolio with counterparty risk.

The most extensive study with regard to the number of simulated risk factors is from *Barnhill* and *Maxwell* (2002). They simulate the risk-free term structure, credit spreads, a foreign exchange rate, and equity market indices, which are all assumed to be correlated. Another extensive study with respect to the modeling of the bank's whole balance sheet (assets, liabilities, and off-balance sheet items) has recently been presented by *Drehmann* et al. (2006). They assess the impact of credit and interest rate risk and their combined effect on the bank's economic value as well on its future earnings and their capital adequacy.

There are also first attempts to build integrated market and credit risk portfolio models for commercial applications, such as the software Algo Credit developed and sold by the risk management firm Algorithmics (see *Iscoe* et al. (1999)).

Examples of the top-down approach are from *Ward* and *Lee* (2002), *Dimakos* and *Aas* (2004), and *Rosenberg* and *Schuermann* (2006). *Dimakos* and *Aas* (2004) apply the copula approach together with some specific (in)dependence assumptions for the aggregation of market, credit and operational risk.[3] *Rosenberg* and *Schuermann* (2006) deal with the aggregation of market, credit and operational risk of a typical large, internationally active bank. They analyze the sensitivity of the aggregate VaR and expected shortfall estimates with respect to the chosen inter-risk correlations and copula functions as well as the given business mix. Furthermore, they compare the aggregate risk estimates resulting from an application of the copula approach with those computed with heuristics used in practice. *Kuritzkes* et al. (2003) discuss and empirically examine risk diversification issues resulting from risk aggregation within financial conglomerates, whereby they also consider the regulator's point of view. Finally, using a normal copula, *Ward* and *Lee* (2002) apply the copula approach for risk aggregation in an insurance company.

An approach, which does not fit entirely neither into the top-down approach nor into the bottom-up approach (as understood in this paper), is from *Alexander* and *Pezier* (2003). They suggest to explain the profit and loss distribution of each business unit by a linear regression model where changes in various risk factors (e.g., risk-free interest rates, credit spreads, equity indices, or implied volatilities) until the desired risk horizon are the explaining factors. From these linear regression models, the standard deviation of the aggregate profit and loss is computed and finally multiplied with a scaling

[3] Later, this work has been significantly extended by *Aas* et al. (2007), where ideas of top-down and bottom-up approaches are mixed.

factor to transform this standard deviation into an economic capital estimate. However, this scaling factor has to be determined by Monte Carlo simulations.

The main contribution of this paper to the literature is that it is the first which directly compares the economic capital requirements based on the bottom-up and the top-down approach. For this, we restrict ourselves to two risk types, market risk (interest rate and credit spread risk) and credit risk (transition and default risk), and we restrict ourselves to the risk measurement of banking book instruments only. Obviously, it would be preferable to consider further risk types, such as operational risk. However, it would be rather difficult to integrate operational risk into a bottom-up approach (actually, the author is not aware of any such an approach) so that a comparison between the bottom-up and the top-down approach would not be possible. Furthermore, it would be preferable to extend the analysis to trading book instruments. However, measuring different risk types of banking book and trading book instruments simultaneously in a bottom-up approach, would make in necessary to employ a dynamic version of a bottom-up approach because only in dynamic version, changes in the trading book composition, as a bank's reaction to adverse market movements, could be considered for measuring the total risk of both books at the common risk horizon. As this extension would introduce much more complexity, we restrict ourselves to the banking book.[4] Nevertheless, measuring the market and credit risks of the banking book precisely would already be a significant step forward because the volume of the banking book of universal banks is typically large compared to the trading book. The profit and loss distribution of the banking book computed by a potentially more exact bottom-up approach could then enter into a top-down approach which integrates all bank risks. Being able to assess the market and the credit risk of the banking book and the interaction of these risk types is also of crucial importance for fulfilling the requirements of the second pillar of the New Basle Accord (see *Basle Committee of Banking Supervision* (2005)). The second pillar requires that banks have a process for assessing their overall capital adequacy in relation to their risk profile. During the capital assessment process, all material risks faced by the bank should be addressed, including for

4 *Rosenberg* and *Schuermann* (2006) link the credit risk of the banking book and the market risk of the trading book together with the operational risk. However, they only consider a top-down approach.

example interest rate risk in the banking book.[5] However, for identifying the bank's risk profile, it is important to correctly consider the interplay between the various risk types.

3. Model Setup

3.1 Portfolio Composition

For the purpose of the comparison between top-down and bottom-up approaches, a stylized banking book composition is employed. It is assumed that the banking book is exclusively composed of assets and liabilities with fixed interest rates and that the bank pursues a typical strategy of positive term transformation (see figure 1). The credits $n \in \{1,\ldots,N\}$ on the asset side are defaultable and mainly exhibit maturity dates T_n of seven to ten years.[6] All credits are structured as zero coupon bonds with a face value of one Euro and are issued by a different corporate. The liabilities $m \in \{1,\ldots,M\}$ of the bank are also structured as zero coupon bonds, but they are assumed to be non-defaultable.

– insert figure 1 about here –

3.2 Bottom-Up Approach

As a bottom-up approach for measuring the credit and market risk of banking book instruments simultaneously, an extended CreditMetrics model is employed. This extension exhibits correlated interest rate and credit spread risk.[7] The risk horizon of the credit portfolio model is denoted by H . P denotes the real world probability measure. The number of possible credit qualities at the risk horizon

5 Anyway, with the New Basle Accord, the regulatory authorities pay more attention to the interest rate risks in the banking book. Banks will have to report the economic effect of a standardized interest rate shock applied to their banking book. If the loss in the banking book as a consequence of the standardized interest rate shock amounts more than 20% of the tier 1 and tier 2 capital, the bank is qualified as an 'outlier' bank. The consequence is that the regulatory authorities analyze the interest rate risk of this bank more thoroughly. At the end, they can even require that the bank reduces its interest rate exposure or increases its regulatory capital.

6 Such long credit periods can typically be observed in Germany.

7 A similar model has been used by *Grundke* (2005).

is K : one denotes the best rating and K the worst rating, the default state. The conditional probability of migrating from rating class $i \in \{1,...,K-1\}$ to $k \in \{1,\ldots,K\}$ within the risk horizon H is assumed to be:

$$P\left(\eta_H^n = k \middle| \eta_0^n = i, Z = z, X_r = x_r\right) := f_{i,k}(z, x_r)$$

$$= \Phi\left(\frac{R_k^i - \sqrt{\rho_R - \rho_{X_r,R}^2}\, z - \rho_{X_r,R} x_r}{\sqrt{1-\rho_R}}\right) - \Phi\left(\frac{R_{k+1}^i - \sqrt{\rho_R - \rho_{X_r,R}^2}\, z - \rho_{X_r,R} x_r}{\sqrt{1-\rho_R}}\right) \tag{1}$$

where η_0^n and η_H^n, respectively, denotes the rating of obligor $n \in \{1,\ldots,N\}$ in $t = 0$ and at the risk horizon $t = H$, respectively, and $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution. Given an initial rating i, the conditional migration probabilities are not assumed to be obligor-specific. The thresholds R_k^i are derived from a transition matrix $Q = (q_{ik})_{1 \le i \le K-1, 1 \le k \le K}$, whose elements q_{ik} specify the (unconditional) probability that an obligor migrates from the rating grade i to the rating grade k within the risk horizon.[8]

The above specification of the conditional migration probabilities corresponds to defining a two-factor-model for explaining the return R_n on firm n's assets in the CreditMetrics model:

$$R_n = \sqrt{\rho_R - \rho_{X_r,R}^2}\, Z + \rho_{X_r,R} X_r + \sqrt{1-\rho_R}\, \varepsilon_n \qquad (n \in \{1,\ldots,N\}) \tag{2}$$

where Z, X_r, and $\varepsilon_1,\ldots,\varepsilon_N$ are mutually independent, standard normally distributed stochastic variables. The stochastic variables Z and X_r represent systematic credit risk, by which all firms are affected, whereas ε_n stands for idiosyncratic credit risk. An obligor n with current rating i is assumed to be in rating class k at the risk horizon when the realization of R_n lies between the two thresholds R_{k+1}^i and R_k^i, with $R_{k+1}^i < R_k^i$. The specification (2) ensures that the correlation $Corr(R_n, R_m)$ ($n \ne m$) between the asset returns of two different obligors is equal to ρ_R. The correlation $Corr(R_n, X_r)$ between the asset returns and the factor X_r is $\rho_{X_r,R}$. As X_r is also the

8 For details concerning this procedure, see *Gupton* et al. (1997, pp. 85).

random variable which drives the term structure of risk-free interest rates (see (4) in the following), $\rho_{X_r,R}$ is the correlation between the asset returns and the risk-free interest rates.

For simplicity, the stochastic evolution of the term structure of risk-free interest rates is modeled by the approach of *Vasicek* (1977).[9] Thus, the risk-free short rate is modeled as a mean-reverting Ornstein-Uhlenbeck process:

$$dr(t) = \kappa(\theta - r(t))\,dt + \sigma_r dW_r(t) \tag{3}$$

where $\kappa, \theta, \sigma_r \in \mathbb{R}_+$ are constants, and $W_r(t)$ is a standard Brownian motion under P. The solution of the stochastic differential equation (3) is:

$$r(t) = \underbrace{\theta + (r(t-1) - \theta)e^{-\kappa}}_{=E^P[r(t)]} + \sqrt{\frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa}\right)}X_r \tag{4}$$

where $X_r \sim N(0,1)$. As the random variable X_r also enters the definition (1) of the conditional transition probabilities, transition risk and interest rate risk are dependent in this model.

The price of a defaultable zero coupon bond at the risk horizon H, whose issuer n has not already defaulted until H and exhibits the rating $\eta_H^n \in \{1,\ldots,K-1\}$, is given by:

$$p(X_r, S_{\eta_H^n}, H, T_n) = \exp\left(-\left(R(X_r, H, T_n) + S_{\eta_H^n}(H, T_n)\right)\cdot(T_n - H)\right). \tag{5}$$

Here, $R(X_r, H, T_n)$ denotes the stochastic risk-free spot yield for the time interval $[H, T_n]$ calculated from the *Vasicek* (1977) model (see *de Munnik* (1996, p. 71), *Vasicek* (1977, pp. 185)). In the *Vasicek* model, the stochastic risk-free spot yields are linear functions of the single risk factor X_r, which drives the evolution of the whole term structure of interest rates. $S_{\eta_H^n}(H, T_n)$ ($\eta_H^n \in \{1,\ldots,K-1\}$) is the

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9 It is well-known that the *Vasicek* model can produce negative interest rates. However, for empirically estimated parameters, the probability for negative interest rates is usually very small. Actually, the CreditMetrics model could be combined with any other term structure model.

stochastic credit spread of rating class η_H^n for the time interval $[H, T_n]$.[10] The rating-specific credit spreads are assumed to be multivariate normally distributed random variables. This is what *Kiesel* et al. (2003) found for the joint distribution of credit spread changes, at least for longer time horizons such as one year, which are usually employed in the context of risk management for banking book instruments.[11] Furthermore, it is assumed that the interest rate factor X_r is correlated with the credit spreads. For the sake of simplicity, this correlation parameter is set equal to a constant $\rho_{X_r,S}$, regardless of the rating grade or the remaining time to maturity. Besides, it is assumed that the idiosyncratic credit risk factors ε_n ($n \in \{1,\ldots,N\}$) are independent of the rating-specific credit spreads $S_{\eta_H^n}(H,T_n)$ ($\eta_H^n \in \{1,\ldots,K-1\}$) for all considered maturity dates T_n. The price $\tilde{p}(X_r,H,T_n)$ of a default risk-free zero coupon bond is computed by discounting the standardized nominal value only with the stochastic risk-free spot yield $R(X_r,H,T_n)$.

If the issuer n of a zero coupon bond has already defaulted ($\eta_H^n = K$) until the risk horizon H, the value of the bond is set equal to the minimum of a beta-distributed fraction δ_n of the value $\tilde{p}(X_r,H,T_n)$ of a risk-free, but otherwise identical, zero coupon bond and the value of the bond without any rating transition of the obligor:

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10 Actually, $S_{\eta_H^n}(H,T_n)$ is the stochastic *average* credit spread of all obligors in the rating class η_H^n. The gaps between the firm-specific credit spreads and the average credit spread of obligors with the same rating are not modeled, but all issuers are treated as if the credit spread appropriate for them equals the average credit spread of the respective rating grade. This assumption also implies independence between the credit spreads and the idiosyncratic risk factors. Without this assumption, the realized asset return of each obligor would have to be linked to a firm-specific credit spread by a Merton-style firm value model which has to be calibrated for each obligor. This seems not to be adequate for practical purposes.

11 Obviously, the assumption of multivariate normally distributed credit spreads implies the possibility of negative realizations. However, as for the risk-free interest rates, this happens only with a very small probability.

$$p(X_r, \eta_H^n = K, \delta_n, H, T_n) = \min\left\{\delta_n \tilde{p}(X_r, H, T_n); p(X_r, S_{\eta_0^n}, H, T_n)\right\}. \quad (6)$$

This is a modified version of the so-called 'recovery-of-treasury' assumption, which ensures that the recovery payment is never larger than the value of the defaultable bond without a default. The recovery rate is assumed to be independent across issuers and independent from all other stochastic variables of the model.

For pricing the liabilities $l(X_r, S_{\eta_0^{bank}}, H, T_m)$ of the bank, it is assumed that the bank cannot default, but remains in its initial rating grade $\eta_0^{bank} = \text{Aa}$. Thus, only the probability distributions of the risk-free interest rates and the Aa credit spreads are relevant for the pricing of the bank's liabilities.[12]

Finally, the value $\Pi(H)$ of the entire banking book at the risk horizon H, comprising the effects of market and credit risks as measured within the bottom-up approach described above, is:

$$\Pi(H) = \sum_{n=1}^{N} p(X_r, S_{\eta_H^n}, H, T_n) - \sum_{m=1}^{M} l(X_r, S_{\eta_0^{bank}=\text{Aa}}, H, T_m). \quad (7)$$

Accordingly, the absolute loss $L(t)$ of the banking book within a period $(t-1, t]$ is defined as $L(t) = \Pi(t-1) - \Pi(t)$, and the log-loss return is $r_L^{BU}(t) = \ln\left(\Pi(t-1)/\Pi(t)\right)$.

3.3 Top-Down Approach

According to Sklar's Theorem, any joint distribution function $F_{X,Y}(x, y)$ can be written in terms of a copula function $C(u, v)$ and the marginal distribution functions $F_X(x)$ and $F_Y(y)$:[13]

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[12] It does not pose any methodological problems to introduce a varying rating of the bank, which depends on the realized return on the bank's assets. Additional simulations show that, as expected, the necessary amount of economic capital decreases due to this modification because a bad performance of the credit portfolio causes a rating downgrade of the bank and, hence, a reduction of the market value of the bank's liabilities.

[13] Standard references for copulas are *Joe* (1997) and *Nelsen* (1999). For a discussion of financial applications of copulas, see, e.g., *Cherubini* et al. (2004).

$$F_{X,Y}(x,y)=C(F_X(x),F_Y(y))\,. \tag{8}$$

The corresponding density representation is:

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)c(F_X(x),F_Y(y)) \tag{9}$$

where $c(u,v)=(\partial^2/\partial u\partial v)C(u,v)$ is the copula density function, and $f_X(x)$ and $f_Y(y)$ are the marginal density functions. For recovering the copula function of a multivariate distribution $F_{X,Y}(x,y)$, the method of inversion can be applied:

$$C(u,v)=F_{X,Y}(F_X^{-1}(u),F_Y^{-1}(v)) \tag{10}$$

where $F_X^{-1}(x)$ and $F_Y^{-1}(y)$ are the inverse marginal distribution functions. In the context of our top-down approach, $F_X(x)$ and $F_Y(y)$ are the marginal distributions of the market and credit risk loss returns of the banking book, measured on a stand-alone basis. Two copula functions frequently used for risk management and valuation purposes are the normal copula and the *t*-copula, which are also employed in this paper. These two copula functions are also typically used when a top-down approach is implemented in practice.

4. Methodology

In the following, the three-step-procedure of generating time series of market and credit risk loss returns by the bottom-up approach, estimating the marginal distributions and the copula parameters and comparing the resulting loss distributions of the bottom-up and the top-down approach are described in detail. Furthermore, the employed parameterization of the model is explained.

4.1 Parameters

The risk horizon H is set equal to one year. The simulations are done for homogeneous initial ratings $\eta_0 \in \{\text{Aa, Baa}\}$. As typical parameters for the *Vasicek* term structure model, $\kappa = 0.4$ and $\sigma_r = 0.01$ are chosen. The mean level θ and the initial short rate $r(0)$ are set equal to 0.06. As market price of interest rate risk λ a value of 0.5 is taken.[14] As the mean and the standard deviation of the beta-distributed recovery rate, $\mu_\delta = 0.530$ and $\sigma_\delta = 0.2686$ are employed. These are typical values observed for senior unsecured bonds by rating agencies. The one-year transition matrix Q equals the average transition probabilities of all corporates rated by Moody's in the period 1970-2005 (see *Hamilton* et al. (2006, p. 25)).

The means and standard deviations of the multivariate normally distributed rating-specific credit spreads $S_k(H,T_n)$ ($k \in \{1,\ldots,K-1\}$) as well as their correlation parameters are taken from *Kiesel* et al. (2003). They use for estimation daily Bloomberg spread data covering the period April 1991 to April 2001. Unfortunately, they only estimate these parameters for times to maturity of two and five years. The correlations between credit spreads $S_k(H,T_n)$ and $S_k(H,\overline{T}_n)$ with different times to maturity $T_n \neq \overline{T}_n$ are also not estimated by them. Thus, for the purpose of this simulation study, it is assumed that the credit spreads with different times to maturity are perfectly correlated and that the credit spread distributions are identical for all times to maturity.[15] This unique credit spread distribution is based on the average parameter values of the distributions determined by *Kiesel* et al. (2003) for times of maturity of two and five years.

The value of the correlation parameter ρ_R between the asset returns is chosen as 0.1. This value is near the lower boundary of the regulatory interval $[0.08; 0.24]$ defined for corporates in the New Basle Accord and, hence, closer to those values observed in empirical studies about asset return correlations

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14 For example, *Barnhill* and *Maxwell* (2002) estimate a short rate volatility of 0.007, whereas *Lehrbass* (1997) finds $\sigma_r = 0.029$, and *Huang* and *Huang* (2003) even work with $\sigma_r = 0.0468$. With regard to the mean reversion parameter and the market price of interest rate risk, *Lehrbass* finds $\kappa = 1.169$ and absolute values of 0.59, 0.808 and 1.232 for the parameter λ, whereas *Huang* and *Huang* choose $\kappa = 0.226$ and an absolute value of 0.248 for λ.

15 In the *Vasicek* model, the spot yields for different times to maturity are also perfectly correlated.

(see, e.g., *Düllmann* and *Scheule* (2003), *Hahnenstein* (2004), or *Dietsch* and *Petey* (2004)). As a robustness check, also the extreme asset return correlation $\rho_R = 0.4$ is tested.

The correlation parameter $\rho_{X_r,S}$ between the credit spreads and the risk-free interest rates is set equal to -0.2. The empirical evidence hints at a negative relationship between changes in risk-free interest rates and changes in credit spreads (see, e.g., *Duffee* (1998), *Düllmann* et al. (2000), *Kiesel* et al. (2002); opposite results are found by *Neal* et al. (2000)). This observation is in line with theoretical pricing models for credit risks (see, e.g., *Longstaff* and *Schwartz* (1995)). The strength of the correlation depends on the rating grade: the absolute value is larger the lower the rating is. However, for simplicity, this effect is neglected in this simulation study.

With respect to the parameter $\rho_{X_r,R}$, which determines the correlation between the firms' asset returns and the term structure of risk-free interest rates, empirical studies about the ability of firm value models to correctly price credit risks usually assume a negative correlation between asset returns and risk-free interest rates (see, e.g., *Lyden* and *Saraniti* (2000, p. 38) or *Eom* et al. (2004, p. 505)). However, *Kiesel* et al. (2002) find that in years with negative interest rate changes, less rating upgrades take place and VaR values are increased. Their result hints at a positive sign for $\rho_{X_r,R}$, which would also be compatible with $\rho_{X_r,S} < 0$. Due to this uncertainty, various parameters $\rho_{X_r,R} \in \{-0.2, 0, 0.2\}$ are tested as a robustness check.

4.2 Data Generation

The sample data matrix $D = \{r_{L_1}(t), r_{L_2}(t)\}_{t=1}^{T}$ of credit and market risk loss returns of the banking book is simulated by means of the bottom-up approach. The realization of the credit risk loss return $r_{L_1}(t) = \ln\left(\Pi_{\eta_0} / \Pi^{credit}(t)\right)$ at time t is generated by the credit portfolio model without considering market risk, but only the risk of transitions between the rating classes. In this case, the future payments are discounted with those risk-free discount factors that correspond to the initial mean level of the short rate and with those default-risky discount factors which correspond to the expected credit spread discount factors. Thus, fluctuations in the term structure of risk-free interest rates or stochastic credit spreads are not considered for computing the losses. Π_{η_0} denotes the value of the banking book when all obligors are in their initial rating class η_0 and no market risk is considered for discounting.

In contrast, the realization of the market risk loss return $r_{L_2}(t) = \ln\left(\Pi_{\eta_0} / \Pi^{market}(t)\right)$ at time t is generated by only considering market risk, but no transition risk. In this case, it is assumed that all obligors stay in their initial rating class within the time period $(t-1,t]$. The future payments are discounted with the risk-free spot yields and the credit spreads of the respective rating grades, observed in t (for the distributional assumptions of the risk-free short rate and the credit spreads, see section 3.2).

For the simulation of the loss data, it is assumed that at the beginning of each period $(t-1,t]$, the cash flow structure is as presented in figure 1 and that all obligors are in their initial rating class η_0. Thus, the cash flow of the previous period and maybe additional capital are used to recover the cash flow structure and, in particular, to compensate credit losses. In particular, a dynamic deterioration of the credit quality of the banking book is not considered. Furthermore, losses due to a decreasing time to maturity are not considered.

The time period between each sample point t of the data matrix $D = \{r_{L_1}(t), r_{L_2}(t)\}_{t=1}^{T}$ is chosen as one year. Overall, the data of 60 bank years is simulated. Alternatively, quarterly data could be simulated by adjusting the transition matrix and the credit spread distribution properly. One quarter is the frequency with which German banks have to measure, for example, the interest rate risk of their banking book according to the tier 2 requirements of the New Basle Accord (see BaFin (2005)). The data of the simulated 60 bank years would correspond to quarterly data gathered by the bank since 15 years. However, compared to what one finds currently in practice, even 15 years of credit loss history would already be a very long time period.

4.3 Estimation of the Marginal Distributions and the Copula Parameters

For the top-down approach, we need the marginal distributions of the credit and market risk loss returns r_{L_1} and r_{L_2}. There are several possibilities how obtain these marginal distributions. First, based on the simulated sample data matrix $D = \{r_{L_1}(t), r_{L_2}(t)\}_{t=1}^{T}$, the marginal distributions can be estimated parametrically. However, this parametric approach suffers from the problem that we have to choose a distribution family à priori and that misspecified marginal distributions can cause a misspecification of the dependence structure expressed by the copula (see *Fermanian* and *Scaillet* (2005)). For modeling the return of market risk positions, a *t*-distribution or a normal mixture distribution are often used because they reflect the fat tails usually observed for market risk returns. For modeling the loss return of credit risk positions, the beta distribution, the lognormal distribution, the Weibull distribution or the Vasicek distribution have been proposed. Second, empirical marginal distributions for loss returns can be derived from single-risk-type models which exist in most banks.[16]

In this study, we use both approaches for estimating the marginal distributions of the random variables r_{L_1} and r_{L_2}. For the second approach, a large number of bank years (200,000 to 2,000,000) are simulated using the data generating model described in section 3.2. For the parametric estimation of the marginal market risk loss return distribution, a normal distribution is chosen. For the parametric estimation of the marginal credit risk loss return distribution, a beta distribution $\beta_{a,b}(r_{L_1})$ with parameters a and b is taken. As the support of the (standard) beta distribution is the unit interval $[0,1]$, negative realizations of the loss return r_{L_1} cannot be modeled by this choice of the marginal distribution. However, these are possible, in particular for lower initial rating grades, due to the mark-to-market approach of the data generating process for the credit losses. For the later simulation of the credit risk loss returns within the top-down approach, the lower boundary of zero for the possible realizations of the credit risk loss return does not pose any problems because for computing risk

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[16] A third approach, which is not pursued in this paper, would be to derive parametric estimates of the marginal loss return distributions based on single-risk type models.

measures only the right tail of the loss distribution is relevant. Furthermore, due to the usage of the beta distribution, loss returns larger than 100% cannot be simulated, too.[17] Loss returns larger than 100% are possible because the returns are defined as log-returns. Thus, in these cases, the top-down approach with the marginal credit risk loss return parameterized as a beta distribution is likely to underestimate the risk measures.

The parameter $\hat{\rho}$ of the bivariate normal copula and the parameters $(\hat{n}, \hat{\rho})$ of the bivariate *t*-copula can be computed, for example, by maximum likelihood estimation.[18] Basically, the parameters of the marginal distributions and the parameters of the copula, combined in the parameter vector θ, can be estimated simultaneously by the maximum likelihood method. Taking into account the density representation (9), the log-likelihood function is:[19]

$$l(\theta) = \ln\left(\prod_{t=1}^{T} f_{r_{L_1}(t), r_{L_2}(t)}(r_{L_1}(t), r_{L_2}(t); \theta)\right)$$

$$= \sum_{t=1}^{T}\left(\ln\left(\left.\frac{\partial^2 C(u,v;\theta)}{\partial u \partial v}\right|_{(u,v)=(F_{r_{L_1}}(r_{L_1}(t);\theta), F_{r_{L_2}}(r_{L_2}(t);\theta))}\right) + \ln\left(f_{r_{L_1}}(r_{L_1}(t);\theta) f_{r_{L_2}}(r_{L_2}(t);\theta)\right)\right). \quad (11)$$

---

17 For estimating the parameters of the copula functions, in those cases in which the realization of the loss return r_{L_1} is negative, which implies an increase in the value of the credit portfolio on a mark-to-market basis, r_{L_1} is set equal to a small positive number (e.g., 0.00001). Whenever r_{L_1} is larger than one, r_{L_1} is set equal to 0.999999. Besides, it is possible that negative portfolio values (with as well as without integrated market and credit risk) are generated, in particular for low initial ratings and large asset return correlations. As computing a log-return is not possible in these cases, negative portfolio values are not considered in the simulated times series, on which the calibration of the top-down approach depends, and during the simulations of the portfolio value with the bottom-up approach.

18 *Rosenberg* and *Schuermann* (2006) do not estimate the parameters of the copula functions, but instead, for $\hat{\rho}$, they employ the results of other studies and expert interviews and the degree of freedom $\hat{n}$ of the *t*-copula is chosen ad hoc.

19 It is assumed that the copula function does not vary within the data period.

As usual, the maximum likelihood estimator (MLE) can be obtained by maximizing the log-likelihood function (11):

$$\hat{\theta}_{\mathrm{MLE}} = \arg\max_{\theta\in\Theta} l(\theta) \tag{12}$$

where Θ is the parameter space. Of course, to apply (11) and (12), an à priori choice of the type of the marginal distribution and the copula (with unknown parameters) is necessary. To reduce the computational costs for solving the optimization problem (12), which results from the necessity to estimate jointly the parameters of the marginal distributions and the copula, the method of inference functions for margins (IFM) can be applied (see *Joe* (1997, pp. 299), *Cherubini* et al. (2004, pp. 156)). The IFM method is a two-step-method where, first, the parameters θ_1 of the marginal distributions are estimated and, second, given the parameters of the marginal distributions, the parameters θ_2 of the copula are determined. The IFM procedure is used in this paper, whereby the parameters of the marginal distributions are computed by the method of moments. A further possibility for estimating the copula function is the canonical maximum likelihood (CML) estimation. For this method, there is no need to specify the parametric form of the marginal distributions because these are replaced by the empirical marginal distributions. Thus, only the parameters of the copula function have to be estimated by MLE (see *Cherubini* et al. (2004, p. 160)). In the following, this approach is employed when the marginal distributions are estimated non-parametrically based on single-risk-type models.

4.4 Comparison of the Loss Distributions

Next, the loss return distributions r_L^{BU} and r_L^{TD}, respectively, of the banking book over a risk horizon of one year are computed. For this, on one hand, the bottom-up approach, as described in section 3.2, is employed and, on the other hand, the top-down approach with a normal and a *t*-copula is used. For computing the loss return r_L^{TD} within the top-down-approach, the simulated credit risk loss return r_{L_1}

and the simulated market risk loss return r_{L_2} are aggregated in the following way to a total loss return:[20]

$$r_L^{TD} = \ln\left(\frac{\Pi_{\eta_0}}{\Pi_{\eta_0} - \underbrace{(\Pi_{\eta_0} - e^{-r_{L1}}\Pi_{\eta_0})}_{\text{losses due to credit risks}} - \underbrace{(\Pi_{\eta_0} - (\Pi_{\eta_0} - e^{-r_{L2}}\Pi_{\eta_0}))}_{\text{losses due to market risks}}}\right) = \ln\left(\frac{1}{e^{-r_{L1}} + e^{-r_{L2}} - 1}\right). \quad (13)$$

The number of simulations D varies from 200,000 to 2,000,000; the exact number is indicated below each table. To compute, for example, the 1%-percentile of r_L, the generated realizations of these random variables are sorted with a Quicksort-algorithm in ascending order and the $(0.01 \cdot D)^{\text{th}}$ of these sorted values is taken as an estimate of the 1%-percentile. As risk measures, the Value-at-Risk (VaR) and the expected shortfall (ES) corresponding to a confidence level of $p \in \{99\%, 99.9\%, 99.97\%\}$ are computed:

$$P\left(r_L > E[r_L] + \text{VaR}(p)\right) = 1 - p, \quad (14)$$

$$\text{ES}(p) = E^P\left[r_L \middle| r_L > E[r_L] + \text{VaR}(p)\right]. \quad (15)$$

Furthermore, 95%-confidence intervals are computed for these risk measures (see *Glasserman* (2004, p. 491) and *Manistre* and *Hancock* (2005)).

To take into account the uncertainty in the parameter estimates of the marginal distributions of r_{L_1} and r_{L_2} as well as the copula parameters caused by the short time series, the three steps described in the three previous sections (1. data generation, 2. estimation of the marginal distributions and the copula

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20 It might be tempting to define $r_L^{TD} = r_{L_1} + r_{L_2}$. However, this definition would cause a systematic underestimation of the risk measures produced by the top-down approach. This underestimation would be larger the larger the absolute losses are. For example, assume that the initial portfolio value is $\Pi_{\eta_0} = 100$, the simulated portfolio value considering only credit risks is $\Pi^{credit} = 70$, and the simulated portfolio value considering only market risks is $\Pi^{market} = 80$. Then, the above additive definition yields a total loss return $r_{L_1} + r_{L_2} = \ln(100/70) + \ln(100/80) = \ln\left(100^2/(80 \cdot 70)\right) = 0.5798$. However, the correct total loss return defined according to (13) is $r_L^{TD} = \ln\left(100/(100 - 30 - 20)\right) = 0.6931$.

parameters, 3. comparison of the loss return distributions resulting from the top-down and the bottom-up approach) are repeated 200 times for the top-down approach. Doing this, an empirical probability distribution for the parameters and the risk measures of the top-down approach can be computed and compared with the risk measures produced by the bottom-up approach.[21]

5. Results

5.1 Base Case

Table 1 shows the first four moments of the empirical marginal distributions for the market and credit risk loss returns. These are based on 2,000,000 bank years. For the market risk loss return, the skewness and excess kurtosis do not contradict the assumed normal distribution. The credit risk loss return is clearly non-normally distributed. Skewness and excess kurtosis are larger the lower the initial rating and the larger the asset return correlation is.

- insert table 1 about here -

Table 2 shows the risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach. For the top-down approach, a normal copula is used, and the displayed risk measures correspond to the mean of the respective numbers over 200 repetitions. For the initial rating Aa, the fit between the risk measures is quite good: the VaR and ES are slightly underestimated by the top-down approach, but the risk measures of both approaches have the same order of magnitude. However, for the initial rating Baa, this is not true any more: here, we can observe a significant underestimation of the risk measures by the top-down approach, which is larger the higher the confidence level of the risk measure is. Unfortunately,

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[21] Alternatively, estimation risk could be considered based on the asymptotic joint maximum likelihood distribution of the parameters of the marginal distributions and the copula function (see *Hamerle* et al. (2005) and *Hamerle* and *Rösch* (2005)). However, due to the short time series, the asymptotic maximum likelihood distribution of the parameters might differ significantly from the real distribution of the parameters.

especially in credit risk management, risk measures corresponding to high confidence levels (usually larger than 99.9%) are needed.

- insert table 2 about here -

One reason for the underestimation of the risk measures by the top-down approach in the case of an initial rating of Baa might be the fact that the beta distribution cannot produce credit risk loss returns larger than one. This is checked by employing the empirical marginal loss distributions based on 200,000 bank years instead of assuming specific distributions for the market and credit risk loss returns and fitting them to the data. As table 3 shows, using this non-parametric approach for the marginal distributions improves the fit between the risk measures produced by the bottom-up and the top-down approach significantly. Furthermore, the uncertainty of the risk measures of the top-down approach is reduced because the uncertainty about the marginal loss distributions is reduced.

- insert table 3 about here -

Table 4 shows the results when the number of simulations on which the empirical marginal distributions and the computation of the risk measures rely is increased to 2,000,000. For table 4, the estimation of the correlation parameter of the normal copula is also based on 2,000,000 bank years (instead of a repeated estimation based on 60 bank years). Furthermore, the asset return correlation ρ_R and the correlation parameter $\rho_{X_r,R}$ between the asset returns and the risk-free interest rates are varied. As can be seen, for the initial rating Baa, the fit between the risk measures produced by the bottom-up and the top-down approach worsens with increasing correlation between the asset returns and the risk-free interest rates.

- insert table 4 about here -

Table 4 also reveals in which way the estimated correlation parameter $\hat{\rho}_{normal}^{\text{CML}}$ of the normal copula depends on the credit quality of the portfolio and the various input correlation parameters. The largest absolute values for $\left|\hat{\rho}_{normal}^{\text{CML}}\right| \approx 43.67\%$ result from a good credit quality Aa, a low asset return correlation $\rho_R = 0.1$ and large absolute values of the correlation $\rho_{X_r,R}$ between the asset returns and the risk-free interest rates. Positive values for $\hat{\rho}^{\text{CML}}_{normal}$ are only produced by $\rho_{X_r,R} < 0$. For comparison, *Rosenberg* and *Schuermann* (2006) *assume* a benchmark correlation between market and credit risk of 50%. This is the midpoint value reported by other studies (see *Rosenberg* and *Schuermann* (2006, table 7, p. 595)).[22] However, the inter-risk correlation of 50% is intended to be the correlation between the credit loss returns of the banking book and the market risk loss returns of the trading book. In contrast, in this study, $\hat{\rho}^{\text{CML}}_{normal}$ is the correlation between the credit and market risk of banking book instruments only.

5.2 Robustness Checks

In the following, various robustness checks are carried out and further aspects are analyzed.

5.2.1 Estimation Risk for the Bottom-Up Approach

Up to now, we have assumed that the bottom-up model corresponds the real world data generating process and that we know its parameters with certainty. In contrast, for the top-down approach, we have estimation risk for the correlation parameter of the normal copula and the marginal distribution functions. The advantage of this assumption is that we can ensure that the observed differences between the loss distributions produced by the top-down and the bottom-up approach are only due to methodological differences, but not due to estimation and/or model risk of the bottom-up approach. However, in reality, we neither know the data generating process nor its parameters, but for implementation, the bottom-up approach also has to be estimated. That is why it has to be analyzed whether the sensitivity with respect to estimation and/or model risk is different in both approaches.

As risk measurement techniques for each separate risk type are more (for market risk) or less (for credit risk) well developed, we focus in the following on the estimation risk for the correlation

[22] This midpoint value is considerably upward biased due to a large correlation parameter of 80% found in expert interviews. The market and credit risk correlation reported by the two other studies is around 30%.

parameter $\rho_{X_r,R}$, which governs the interplay between interest rate and transition risk in the bottom-up approach. With respect to informations on distributions and parameters that are only relevant for each separate risk type, we assume that these are exact. For each simulated time series of 60 bank years, the parameter $\rho_{X_r,R}$ is estimated by maximum likelihood. For this, the number of transitions (upgrades and downgrades) away from the initial rating are counted per period. Then, $\hat{\rho}_{X_r,R}$ is given by the solution of the following optimization problem (see, similarly, *Gordy* and *Heitfeld* (2002), *Hamerle* et al. (2003), *van Landschoot* (2005)):

$$\max_{\rho_{X_r,R}\in\left[-\sqrt{\rho_R},-\sqrt{\rho_R}\right]} \sum_{t=1}^{60} \ln \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \binom{N_t}{tr_t} f_{\eta_0,\eta_0}(z,x_r)^{(N_t-tr_t)}(1-f_{\eta_0,\eta_0}(z,x_r))^{tr_t}\,\phi(z)\phi(x_r)\,dz\,dx_r \qquad (16)$$

where tr_t is the number of transitions away from the initial rating η_0 in period t, N_t is the number of obligors in period t, $f_{\eta_0,\eta_0}(z,x_r)$ is the conditional probability to stay in the initial rating grade within one period, and $\phi(\cdot)$ is the density function of the standard normal distribution. For computing the above integrals, the Gauss-Legendre integration rule with $n=96$ grid points is applied, and the integration intervals are transformed to the unit interval $[0,1]$.

For the results of table 5, 200 time series of 60 bank years are generated and for each time series, the correlation parameters $\hat{\rho}_{X_r,R}$ and $\hat{\rho}^{\mathrm{CML}}_{normal}$ of the bottom-up and the top-down approach, respectively, are estimated. Based on these estimates, the VaR and ES risk measures are computed by Monte Carlo simulation with the bottom-up and the top-down approach for each simulation run. For the top-down approach, the empirical marginal distributions based on 200,000 bank years are employed. For both approaches, the computed risk measures are influenced by the Monte Carlo simulation error and the estimation error simultaneously. As table 5 shows, the estimator $\hat{\rho}_{X_r,R}$ is biased due to a small sample error. For the initial rating Aa, the uncertainty of the risk measure estimates, measured by their standard deviation and the width of the [5%,95%-percentile] interval, is comparable. However, for the initial rating Baa, the influence of the estimation risk is more pronounced for the top-down approach than for the bottom-up approach.

- insert table 5 about here -

5.2.2 *t*-Copulas for the Top-Down Approach

Table 6 shows the risk measures when a *t*-copula and the empirical marginal distributions are employed for the top-down approach. Due to its tail dependence, it is expected that in the case of a *t*-copula, we have better fit between the risk measures produced by both approaches. Indeed, as can be seen in table 6, choosing the degree of freedom n of the *t*-copula small enough can produce risk measures that are even larger than those of the bottom-up approach. In particular, the choice $n = 15$ yields a quite good fit.

- insert table 6 about here -

However, for the results of table 6, we just *assumed* that a *t*-copula with a specific degree of freedom is the correct copula for describing the dependence between the market and credit risk loss returns. The important question is whether this assumption is also supported by the simulated data. This, however, seems not to be the case. Based on time series of market and credit risk loss returns with length of 50,000 bank years, the parameters $\hat{\rho}_t^{\mathrm{CML}}$ and n of an assumed *t*-copula have been estimated for various scenarios ($\eta_0 \in \{\mathrm{Aa}, \mathrm{Baa}\}$, $\rho_R \in \{0.1, 0.4\}$, and $\rho_{X_r,R} \in \{-0.2, 0, 0.2\}$). For most scenarios, the estimated degree of freedom is larger than 300 and in no case it is smaller than 100 (without table). This indicates that the *t*-copula is not an adequate modeling approach because it is not backed by the data.

5.2.3 Goodness-of-Fit Test for the Copula Function of the Top-Down Approach Based on Rosenblatt's Transformation

The results of the previous section show that a correct identification of the copula function is of essential importance for the risk measures produced by the top-down approach. In practical applications, the copula function itself is just *assumed* to be the correct one and often even the parameters of this assumed copula function are not estimated on time series data, but are based on so-called expert views (an exception is for example *Cech* and *Fortin* (2006)). However, doing this, nearly any desired risk measure can be produced. Thus, an important question is whether it is possible to infer from given time series data of loss returns the correct copula function or, at least, to reject the null hypothesis of specific copula assumptions. This means that we have to do goodness-of-fit (GoF) tests for the copula function employed by the top-down approach.

GoF tests are not extensively discussed in the literature, but recently some contributions emerged (see, e.g., *Malevergne* and *Sornette* (2003), *Breymann* et al. (2003), *Chen* et al. (2004), *Dobrić* and *Schmid* (2005), *Fermanian* (2005), *Genest* et al. (2006), *Kole* et al. (2007)). From the proposed GoF tests, we choose a test based on Rosenblatt's transformation (see for the following *Dobrić* and *Schmid* (2007)). The Rosenblatt transformation $S(X,Y)$ of two random variables X and Y is defined as (see *Rosenblatt* (1952), *Dobrić* and *Schmid* (2007)):

$$S(X,Y)=\left[\Phi^{-1}\left(F_X(X)\right)\right]^2+\left[\Phi^{-1}\left(C\left(F_Y(Y)\middle|F_X(X)\right)\right)\right]^2 \tag{17}$$

where $C\left(F_Y(Y)\middle|F_X(X)\right)=\partial C(u,v)/\partial u\big|_{(u,v)=(F_X(X),F_Y(Y))}$ is the conditional distribution function of $v=F_Y(Y)$ given $u=F_X(X)$ and $C(u,v)$ is the copula function describing the dependence between the random variables X and Y (see (10)). As the random variables $Z_1=U=F_X(X)$ and $Z_2=C\left(V\middle|U\right)=C\left(F_Y(Y)\middle|F_X(X)\right)$ are independent and uniformly distributed on $[0,1]$, the Rosenblatt transformation $S(X,Y)$ is chi squared distributed with two degrees of freedom. Thus, the validity of the null hypothesis of interest H_0: "(X,Y) has copula $C(u,v)$" implies the validity of the null hypothesis H_0^*: "$S(X,Y)$ is χ_2^2-distributed" and H_0 can be rejected if H_0^* is rejected. For testing the null hypothesis H_0^*, *Dobrić* and *Schmid* (2007) propose to employ the Anderson Darling (AD) test statistic due to its very good power properties:

$$AD=-N-\frac{1}{N}\sum_{j=1}^{N}(2j-1)\cdot\left[\ln\left(G\left(S_{(j)}\right)\right)+\ln\left(1-G\left(S_{(N-j+1)}\right)\right)\right] \tag{18}$$

where (x_1,y_1), … , (x_N,y_N) is a random sample from (X,Y) and $S_{(1)}\le\ldots\le S_{(N)}$ are the increasingly ordered Rosenblatt transformations $S(x_j,y_j)$ ($j\in\{1,\ldots,N\}$) of this random sample. G is the distribution function of a chi squared distributed random variable with two degrees of freedom. The test statistic AD could then be compared against the critical values of its theoretical distribution.

However, *Dobrić* and *Schmid* (2007) point at two problems which appear when this approach is applied for testing for a specific copula function. First, the parameter vector of the assumed copula has to be estimated and used for computing the Rosenblatt transformations $S(x_j, y_j)$ ($j \in \{1, \ldots, N\}$). Second, the true marginal distribution functions $F_X(x)$ and $F_Y(y)$ are typically not known in empirical applications, but have to be substituted by their empirical counterparts when computing the Rosenblatt transformations:

$$\hat{F}_X(x) = \frac{1}{N+1}\sum_{j=1}^{N} 1_{\{X_j \le x\}} \text{ and } \hat{F}_Y(y) = \frac{1}{N+1}\sum_{j=1}^{N} 1_{\{Y_j \le y\}} \tag{19}$$

By means of Monte Carlos simulations, *Dobrić* and *Schmid* (2007) show that in particular as a consequence of this latter fact the true error probability of the first kind of this test is much smaller than the prescribed level. Furthermore, they show for selected alternatives that employing the empirical marginal distribution functions has a negative effect on the power of the test, i.e. the probability of rejecting the null hypothesis. Therefore, *Dobrić* and *Schmid* (2007) propose a bootstrap algorithm (see Appendix) for determining the distribution function of the AD test statistic under the null hypothesis when the empirical marginal distribution functions are employed. This distribution can significantly differ from the theoretical distribution of the AD test statistic that results from the usage of the true marginal distributions. Thus, the critical values can also be significantly different.[23]

For testing whether it is possible to reject the null hypothesis of a specific copula function with only 60 data points, we use in the following the bootstrap and the non-bootstrap version of the above GoF test. When the non-bootstrap version is applied, the empirical marginal distributions of the market and credit risk loss returns based on 200,000 bank years are employed for computing the Rosenblatt transformations and, hence, the AD test statistic for the simulated time series of 60 bank years. These empirical marginal distributions are considered to be sufficiently exact to justify the usage of the theoretical distribution of the AD test statistic (see *Marsaglia* and *Marsaglia* (2004)). When the bootstrap version is applied, the empirical marginal distributions are based on the simulated time series of only 60 bank years, which is also used for estimating the parameters of the copula function.

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[23] This is also noted by *Chen* et al. (2004).

Figure 2 shows the results for testing the null hypothesis of a *t*-copula with varying degrees of freedom n. The median *p*-values for testing the null hypothesis of a normal copula, which are not shown in figure 2, are between 44.79% and 49.90%. As can be seen, neither the bootstrap version nor the non-bootstrap version of the GoF test has the power to reject the null hypothesis of normal copula and a *t*-copula, respectively. The *p*-values are much too large for a reasonable rejection of these two null hypothesis. An exception is the *t*-copula with one degree of freedom which can be rejected at a reasonable confidence level. Thus, based on time series with only 60 data points and based on the above GoF test, it can not be differentiated between a normal and a *t*-copula. As a consequence, non-verifiable *assumptions* on the adequate copula function are unavoidable and a considerable amount of model risk remains.

- insert figure 2 about here -

5.2.4 Modified Definition of the Market and Credit Risk Loss Returns

Another important reason for the more or less pronounced underestimation of the risk measures by the top-down approach (see tables 3 and 4) might the definition of the market and credit risk loss returns and, hence, the quality of the data on which the top-down approach is calibrated. The integrated view of the bottom-up approach can reproduce the stochastic dependency between the credit quality transitions η_H^n and the distribution of the credit spreads $S_{\eta_H^n}(H,T)$ at the risk horizon. Thus, it is possible to capture situations in which obligors are downgraded and, simultaneously, the realization of the credit spread of the respective rating class, in which they are downgraded, is large. In contrast, with the loss definitions used up to now, the market and credit risk loss data, on which the calibration of the top-down approach depends, cannot reflect this dependence because for the simulation of the market loss data, it is assumed that all obligors remain in their initial rating class within the time period $(t-1,t]$. As a consequence, the top-down approach cannot reproduce extreme losses due to simultaneous downgrades and adverse movements of credit spreads. Furthermore, it might be suspected that the excess kurtosis and the skewness of the marginal market risk loss return are too low because the larger means and volatilities of the credit spreads of lower rating grades and the smaller means and volatilities of the better rating grades, respectively, are not considered.

Thus, as further robustness check, the influence of the employed definition of the market and credit risk loss returns used by the different bank divisions is analyzed next. For the results of tables 7 and 8, the process that generates the data on which the top-down approach is calibrated has been modified. First (for table 7), the credit risk loss return only considers default risk, whereas the market risk loss return contains transition risk (without defaults), credit spread risk and interest rate risk. Thus, it is assumed that the market risk division tracks the rating of the obligors until the end of the time period $(t-1,t]$ and that the credit risk division only informs about losses due to a default. Doing this, the market risk division can also report about losses resulting from a simultaneous downgrade of the obligor and a large realization of the credit spread of the respective rating class in which the obligor is downgraded. As table 7 shows, the difference between the risk measures for an initial rating of Baa and an asset return correlation of 10% produced by the top-down and the bottom-up approach becomes smaller with this modified data generation assumption, in particular for $\rho_{X_r,R} \in \{0, 0.2\}$ (compare with table 4). However, when the default probability for initially Baa rated obligors is increased to 1% and the other transition probabilities are proportionally adjusted, there is still a considerable underestimation of the risk measures by the top-down approach.

Second (for table 8), credit risk losses are defined as losses due to rating transitions and defaults, whereas market risk losses are defined as losses due to movements of the risk-free interest rates and due to deviations of the rating-specific credit spread from its mean in the rating class of the obligor at the end of a period. With these loss definitions, the fit between the risk measures is further improved for the initial rating Aa (compare with table 4). For the initial rating Baa, an improvement compared to the results that the two previously employed loss definitions yield can only be observed for $\rho_{X_r,R} = -0.2$ (compare with tables 4 and 7). However, for the initial rating Baa and the increased asset return correlation $\rho_R = 0.4$, the fit between the risk measures is improved for all tested correlations

$\rho_{X_r,R}$ between the asset returns and the risk-free interest rates, compared with the results of table 4 (without table). These results demonstrate that the employed loss definitions are important for the accuracy of the top-down approach, but, compared for example with the significance of the correctness of the marginal distributions or the employed copula function, the importance seems to be of second order.

- insert tables 7 and 8 about here -

6. Conclusions

In this paper, two sophisticated approaches of risk aggregation, the top-down and the bottom-up approach, are compared. It can be observed that in specific situations, for example for portfolios with lower credit qualities, the necessary amount of economic capital can be significantly underestimated by the top-down approach. Furthermore, the accuracy of the marginal loss distributions, the employed copula function, and the loss definitions have an impact on the performance of the top-down approach. Unfortunately, given limited access to time series data of market and credit risk loss returns, it is rather difficult to decide which copula function an adequate modelling approach for reality is.

However, the analysis of the differences between the loss predictions produced by the top-down and the bottom-up approach, which is initiated in this paper, is certainly not at its end, but much remains to be done. Let us only mention two examples: first, typically, banks manage their market risk actively. Thus, when adverse market movements become obvious, banks try to restrict market losses, for example by the usage of derivatives. To consider this bank behaviour for the comparison of top-down and bottom-up approaches, dynamic models are needed, which reflect the evolution of risk factors and portfolio compositions through time. With such models at hand, the analysis could also be extended to encompass the trading book for which a frequent portfolio restructuring is typical. Second, the challenging question could be tackled whether it is possible to identify a family of copula functions that is parametrically parsimonious, and still able to fit the bottom-up model over a wide range of portfolios. In this paper, only results for the normal and the *t*-copula are presented because these copula functions are frequently used when a top-down approach is implemented in practice.

Appendix: GoF-Test for Copulas Based on Rosenblatt's Transformation

The following bootstrap algorithm for computing the distribution function of the AD test statistic under the null hypothesis when the empirical marginal distribution functions are employed has been proposed by *Dobrić* and *Schmid* (2007):

1. Based on the originally observed random sample (x_1, y_1), … , (x_N, y_N), estimate the parameter vector $\hat{\theta}$ of the parametric family of copulas C_θ, which is tested in the null hypothesis H_0: "(X,Y) has copula $C_\theta(u,v)$".

2. Repeat for $s = 1, \ldots, S_B$ where S_B is the number of bootstrap simulations:

   2a. Generate i.i.d. observations $(x_1^{(s)}, y_1^{(s)})$, … , $(x_N^{(s)}, y_N^{(s)})$ from $C_{\hat{\theta}}$.

   2b. Estimate the parameter vector $\hat{\theta}^{(s)}$ from $(x_1^{(s)}, y_1^{(s)})$, … , $(x_N^{(s)}, y_N^{(s)})$ and compute $\hat{S}^{(s)}(x_1^{(s)}, y_1^{(s)})$, … , $\hat{S}^{(s)}(x_N^{(s)}, y_N^{(s)})$ based on the empirical distribution functions $\hat{F}_X(x)$ and $\hat{F}_Y(y)$. The Rosenblatt transformations $\hat{S}^{(s)}(x_1^{(s)}, y_1^{(s)})$, … , $\hat{S}^{(s)}(x_N^{(s)}, y_N^{(s)})$ are then used to compute the value $AD^{(s)}$ of the AD test statistic.

3. Based on $AD^{(1)}, \ldots, AD^{(S_B)}$, the empirical distribution function of the AD test statistic under the null hypothesis can be computed. The critical value corresponding to an error probability of the first kind α equals the $(1-\alpha)$-percentile of empirical distribution function of the AD test statistic. The null hypothesis H_0: "(X,Y) has copula $C_\theta(u,v)$" is rejected if the AD test statistic computed from the originally observed random sample (x_1, y_1), … , (x_N, y_N) is larger than that critical value.

When C_θ is a normal copula, only the correlation parameter ρ has to be estimated. This can be done based on the estimated value $\hat{\rho}_{Sp}$ of Spearman's coefficient of correlation and the relationship $\hat{\rho} = 2\sin\left(\pi\hat{\rho}_{Sp}/6\right)$ (see *McNeil* et al. (2005, p. 230)), or, alternatively, by ML. When C_θ is a *t*-copula, ρ can be estimated by $\hat{\rho} = \sin\left(\pi\hat{\tau}/2\right)$ where $\hat{\tau}$ is the estimated value of Kendall's τ (see *McNeil* et al. (2005, p. 231)).

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Tables

Table 1:

Table 1 shows the first four moments of the empirical marginal market and credit risk loss returns. These are based on 2,000,000 simulated bank years. Parameters: $\rho_{X_r,R} = -0.2$, $\rho_{X_r,S} = -0.2$.

credit risk loss return				
	initial rating Aa		initial rating Baa	
	$\rho_R = 0.1$	$\rho_R = 0.4$	$\rho_R = 0.1$	$\rho_R = 0.4$
mean	1.02%	1.04%	6.01%	6.99%
standard deviation	0.80%	2.04%	7.36%	22.82%
skewness	1.92	6.93	2.45	8.71
excess kurtosis	6.81	104.49	12.27	123.99

market risk loss return				
	initial rating Aa		initial rating Baa	
	$\rho_R = 0.1$	$\rho_R = 0.4$	$\rho_R = 0.1$	$\rho_R = 0.4$
mean	0.10%	0.11%	2.40%	2.21%
standard deviation	5.52%	5.55%	21.70%	21.50%
skewness	0.09	0.08	0.60	0.62
excess kurtosis	0.01	0.02	0.92	0.85

Table 2:

Table 2 shows risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach when the marginal loss distributions are estimated parametrically. For the top-down approach, a normal copula is used, and the displayed risk measures correspond to the mean of the respective numbers over 200 repetitions. For each repetition, the estimation of the parameters of the top-down approach is based on the data of 60 bank years (corresponds to quarterly data of 15 years) and the computation of the risk measures is based on 200,000 simulated bank years; this is also the number of simulations done for the bottom-up approach. VaR_p : Value-at-Risk for the confidence level p, ES_p : Expected Shortfall for the confidence level p, $\hat{\rho}_{normal}^{\text{IFM}}$: mean estimate of the correlation parameter of the normal copula over 200 repetitions computed by the method of inference functions for margins (IFM), std: standard deviation over 200 repetitions, [5%-perc, 95%-perc]: 5%-perentile and 95%-percentile, respectively, of the simulated risk measures over 200 repetitions. Parameters: $\rho_R = 0.1$, $\rho_{X_r,S} = -0.2$, $\rho_{X_r,R} = -0.2$, all other parameters are as described in section 4.1.

		Bottom-Up Approach		Top-Down Approach		
initial rating Aa						
mean		1.10%		1.08%		
standard deviation		6.03%		5.90%		
	p		95%-confidence interval		std	[5%-perc, 95%-perc]
VaR_p	99%	14.90%	[14.79%,15.01%]	14.10%	1.41%	[11.69%,16.24%]
	99.9%	21.39%	[20.95%,21.79%]	19.08%	1.99%	[15.84%,22.09%]
	99.97%	23.94%	[23.14%,24.72%]	21.41%	2.29%	[17.75%,24.94%]
ES_p	99%	18.78%	[17.43%,20.13%]	17.39%	1.94%	[14.00%,20.34%]
	99.9%	24.65%	[18.99%,30.31%]	22.09%	2.48%	[17.89%,25.90%]
				$\hat{\rho}_{normal}^{\text{IFM}}$		
				39.22%	10.72%	[21.31%,57.57%]
initial rating Baa						
mean		8.71%		8.96%		
standard deviation		25.88%		24.38%		
	p		95%-confidence interval		std	[5%-perc, 95%-perc]
VaR_p	99%	75.66%	[74.87%,77.08%]	63.51%	9.80%	[50.44%,83.31%]
	99.9%	139.75%	[130.54%,142.56%]	97.69%	21.05%	[73.56%,142.50%]
	99.97%	180.38%	[159.31%,206.14%]	117.70%	29.58%	[85.54%,174.55%]
ES_p	99%	110.07%	[103.45%,116.68%]	87.22%	16.24%	[65.62%,120.87%]
	99.9%	185.59%	[152.34%,218.64%]	123.11%	30.24%	[89.22%,184.83%]
				$\hat{\rho}_{normal}^{\text{IFM}}$		
				3.02%	7.41%	[-9.33%,14.28%]

Table 3:

Table 3 shows risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach when the empirical marginal loss distributions based on 200,000 bank years are employed. For the top-down approach, a normal copula is used, and the displayed risk measures correspond to the mean of the respective numbers over 200 repetitions. For each repetition, the estimation of the parameter of the top-down approach is based on the data of 60 bank years (corresponds to quarterly data of 15 years) and the computation of the risk measures is based on 200,000 simulated bank years; this is also the number of simulations done for the bottom-up approach. VaR_p : Value-at-Risk for the confidence level p, ES_p : Expected Shortfall for the confidence level p, $\hat{\rho}^{\text{CML}}_{normal}$: mean estimate of the correlation parameter of the normal copula over 200 repetitions computed by canonical maximum likelihood (CML) estimation, std: standard deviation over 200 repetitions, [5%-perc, 95%-perc]: 5%-perentile and 95%-percentile, respectively, of the simulated risk measures over 200 repetitions. Parameters: $\rho_R = 0.1$, $\rho_{X_r,S} = -0.2$, $\rho_{X_r,R} = -0.2$, all other parameters are as described in section 4.1.

		Bottom-Up Approach		**Top-Down Approach**		
initial rating Aa						
mean		1.10%		1.14%		
standard deviation		6.03%		5.95%		
	p		95%-confidence interval		std	[5%-perc, 95%-perc]
VaR_p	99%	14.90%	[14.79%,15.01%]	14.61%	0.28%	[14.16%,15.12%]
	99.9%	21.39%	[20.95%,21.79%]	20.31%	0.51%	[19.44%,21.24%]
	99.97%	23.94%	[23.14%,24.72%]	23.00%	0.67%	[22.02%,24.09%]
ES_p	99%	18.78%	[17.43%,20.13%]	18.23%	0.37%	[17.59%,18.91%]
	99.9%	24.65%	[18.99%,30.31%]	23.65%	0.64%	[22.60%,24.79%]
				$\hat{\rho}^{\text{CML}}_{normal}$		
				42.10%	9.21%	[26.56%,58.82%]
initial rating Baa						
mean		8.71%		8.94%		
standard deviation		25.88%		25.70%		
	p		95%-confidence interval		std	[5%-perc, 95%-perc]
VaR_p	99%	75.66%	[74.87%,77.08%]	77.24%	4.89%	[69.57%,85.73%]
	99.9%	139.75%	[130.54%,142.56%]	131.43%	14.36%	[108.98%,155.78%]
	99.97%	192.20%	[159.31%,206.14%]	171.66%	24.24%	[132.86%,213.84%]
ES_p	99%	110.07%	[103.45%,116.68%]	109.92%	9.11%	[95.40%,125.55%]
	99.9%	185.59%	[152.34%,218.64%]	178.61%	23.75%	[141.86%,220.33%]
				$\hat{\rho}^{\text{CML}}_{normal}$		
				7.83%	12.35%	[-12.43%,28.75%]

Table 4:

Table 4 shows risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach. For the top-down approach, a normal copula is used, and the empirical marginal distribution functions based on 2,000,000 bank years are employed. The estimation of the parameter of the top-down approach is based on the data of 2,000,000 bank years. The computation of the risk measures is also based on 2,000,000 simulated bank years for both approaches. VaR_p : Value-at-Risk for the confidence level p, ES_p : Expected Shortfall for the confidence level p, $\hat{\rho}_{normal}^{\text{CML}}$: estimate of the correlation parameter of the normal copula computed by canonical maximum likelihood (CML) estimation, Parameters: $\rho_{X_r,S} = -0.2$, all other parameters are as described in section 4.1 or indicated in table 4.

		Bottom-Up Approach			**Top-Down Approach**		
initial rating Aa, $\rho_R = 0.1$		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		1.10%	1.16%	1.18%	1.13%	1.16%	1.13%
standard deviation		6.03%	5.77%	5.45%	5.95%	5.67%	5.35%
	p						
VaR_p	0.99	14.88%	14.02%	12.99%	14.57%	13.76%	12.87%
	0.999	21.34%	19.15%	17.67%	20.35%	18.65%	17.48%
	0.9997	23.75%	21.64%	19.73%	23.12%	20.98%	19.57%
ES_p	0.99	18.73%	17.50%	16.17%	18.17%	17.09%	16.00%
	0.999	24.51%	22.45%	20.53%	23.72%	21.60%	20.21%
$\hat{\rho}_{normal}^{\text{CML}}$					43.16%	-0.09%	-43.67%
initial rating Aa, $\rho_R = 0.4$		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		1.14%	1.19%	1.19%	1.17%	1.18%	1.16%
standard deviation		6.44%	6.21%	5.85%	6.31%	6.07%	5.83%
	p						
VaR_p	0.99	17.42%	15.84%	15.05%	16.47%	15.20%	13.81%
	0.999	30.16%	30.69%	24.03%	28.01%	25.51%	20.66%
	0.9997	40.53%	44.40%	29.58%	39.54%	36.75%	28.77%
ES_p	0.99	24.27%	23.06%	20.19%	22.70%	21.19%	19.11%
	0.999	41.99%	42.42%	32.60%	39.39%	40.04%	37.73%
$\hat{\rho}_{normal}^{\text{CML}}$					22.42%	0.30%	-22.05%
initial rating Baa, $\rho_R = 0.1$		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		8.72%	8.77%	9.05%	8.93%	8.61%	8.58%
standard deviation		25.89%	25.67%	25.31%	25.55%	24.62%	24.21%
	p						
VaR_p	0.99	75.82%	75.83%	74.11%	74.70%	71.94%	69.81%
	0.999	139.73%	142.49%	127.66%	128.09%	120.12%	116.39%
	0.9997	196.20%	191.08%	202.38%	168.56%	152.18%	145.69%
ES_p	0.99	110.38%	112.92%	107.54%	107.02%	101.52%	97.75%
	0.999	187.93%	197.84%	186.62%	175.96%	159.30%	151.79%
$\hat{\rho}_{normal}^{\text{CML}}$					7.65%	-0.29%	-7.11%

Table 4 (continued)

initial rating Baa, $\rho_R = 0.4$		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		9.10%	8.76%	9.80%	9.25%	8.81%	9.35%
standard deviation		33.42%	31.92%	33.50%	32.43%	31.36%	31.36%
	p						
VaR_p	0.99	116.87%	114.57%	118.10%	109.93%	107.67%	105.76%
	0.999	315.18%	249.34%	305.75%	273.50%	256.56%	255.02%
	0.9997	423.43%	369.21%	406.19%	385.49%	357.58%	367.07%
ES_p	0.99	197.89%	185.65%	203.67%	186.49%	178.03%	177.10%
	0.999	402.66%	341.58%	385.62%	378.58%	351.34%	358.80%
$\hat{\rho}^{\mathrm{CML}}_{normal}$					3.30%	-0.96%	-3.33%

Table 5:

Table 5 shows risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach when the empirical marginal loss distributions based on 200,000 bank years are employed. For the top-down approach, a normal copula is used, and the displayed risk measures correspond to the mean of the respective numbers over 200 repetitions. For each repetition, the estimation of the parameter of the top-down approach is based on the data of 60 bank years (corresponds to quarterly data of 15 years) and the computation of the risk measures is based on 200,000 simulated bank years; this is also the number of simulations done for the bottom-up approach. For the bottom-up approach, the displayed risk measures also correspond to the mean of the respective numbers over 200 repetitions. For each repetition, the estimation of the correlation $\rho_{X_r,R}$ between the asset returns and the risk-free interest rates is based on the number of transitions away from the initial rating in each of 60 bank years. VaR_p : Value-at-Risk for the confidence level p, ES_p : Expected Shortfall for the confidence level p, $\hat{\rho}_{normal}^{\text{CML}}$: mean estimate of the correlation parameter of the normal copula over 200 repetitions computed by canonical maximum likelihood (CML) estimation, $\hat{\rho}_{X_r,R}$: mean estimate of the correlation between the asset returns and the risk-free interest rates over 200 repetitions, std: standard deviation over 200 repetitions, [5%-perc, 95%-perc]: 5%-perentile and 95%-percentile, respectively, of the simulated risk measures over 200 repetitions. Parameters: $\rho_R = 0.1$, $\rho_{X_r,S} = -0.2$, $\rho_{X_r,R} = -0.2$, all other parameters are as described in section 4.1.

		Bottom-Up Approach			Top-Down Approach		
			std	[5%-perc, 95%-perc]		std	[5%-perc, 95%-perc]
initial rating Aa							
mean		1.13%	0.02%	[1.09%,1.17%]	1.15%	0.02%	[1.13%,1.17%]
standard deviation		6.04%	0.13%	[5.77%,6.16%]	5.94%	0.07%	[5.80%,6.05%]
	p						
VaR_p	99%	15.14%	0.56%	[14.00%,15.70%]	14.60%	0.30%	[14.04%,15.02%]
	99.9%	21.25%	1.05%	[19.15%,22.23%]	20.27%	0.56%	[19.32%,21.16%]
	99.97%	24.10%	1.27%	[21.54%,25.99%]	22.98%	0.74%	[21.72%,24.19%]
ES_p	99%	18.98%	0.76%	[17.49%,19.57%]	18.22%	0.41%	[17.47%,18.86%]
	99.9%	24.85%	1.22%	[22.39%,26.36%]	23.62%	0.71%	[22.38%,24.79%]
		$\hat{\rho}_{X_r,R}$			$\hat{\rho}_{normal}^{\text{CML}}$		
		-17.72%	8.37%	[-28%,0%]	42.73%	10.04%	[23.05%,56.77%]
initial rating Baa							
mean		8.65%	0.16%	[8.45%,8.92%]	8.91%	0.28%	[8.43%,9.35%]
standard deviation		25.83%	0.30%	[25.44%,26.34%]	25.48%	1.41%	[22.97%,27.72%]
	p						
VaR_p	99%	77.12%	1.68%	[74.57%,79.82%]	74.67%	5.82%	[64.58%,83.75%]
	99.9%	139.03%	10.89%	[125.43%,157.05%]	129.51%	16.98%	[102.74%,161.74%]
	99.97%	181.53%	15.98%	[162.11%,209.11%]	170.29%	28.20%	[131.38%,219.86%]
ES_p	99%	111.03%	3.78%	[105.68%,117.60%]	107.20%	10.77%	[89.94%,125.01%]
	99.9%	180.60%	13.23%	[162.36%,199.89%]	176.70%	27.92%	[134.23%,224.47%]
		$\hat{\rho}_{X_r,R}$			$\hat{\rho}_{normal}^{\text{CML}}$		
		-19.39%	6.29%	[-29%,-12%]	6.91%	14.67%	[-20.58%,29.94%]

Table 6:

Table 6 shows risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach. For the top-down approach, a *t*-copula with *n* degrees of freedom is used, and the empirical marginal distribution functions based on 2,000,000 bank years are employed. The estimation of the correlation parameter of the *t*-copula is based on the data of 2,000,000 bank years. The computation of the risk measures is based on 2,000,000 simulated bank years for both approaches. VaR_p: Value-at-Risk for the confidence level p, ES_p: Expected Shortfall for the confidence level p, Parameters: $\rho_R = 0.1$, $\rho_{X_r,R} = -0.2$, $\rho_{X_r,S} = -0.2$, all other parameters are as described in section 4.1 or indicated in table 6.

		Bottom-Up Approach	Top-Down Approach		
initial rating Aa			$n=5$	$n=15$	$n=25$
mean		1.10%	1.14%	1.14%	1.14%
standard deviation		6.03%	5.96%	5.95%	5.95%
	p				
VaR_p	0.99	14.88%	14.84%	14.65%	14.63%
	0.999	21.27%	21.41%	20.65%	20.53%
	0.9997	23.75%	24.81%	23.75%	23.51%
ES_p	0.99	18.73%	18.81%	18.39%	18.33%
	0.999	24.50%	25.42%	24.32%	24.14%
initial rating Baa			$n=5$	$n=15$	$n=25$
mean		8.72%	8.93%	8.90%	8.94%
standard deviation		25.89%	26.35%	25.76%	25.68%
	p				
VaR_p	0.99	75.65%	79.86%	77.46%	75.47%
	0.999	139.74%	161.32%	140.51%	135.00%
	0.9997	196.20%	233.57%	192.89%	182.97%
ES_p	0.99	110.34%	123.57%	113.68%	110.47%
	0.999	187.76%	237.93%	198.07%	190.14%

Table 7:

Table 7 shows risk measures for the banking book loss returns computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach. For the top-down approach, a normal copula is used, and the empirical marginal distribution functions based on 2,000,000 bank years are employed. The estimation of the parameter of the top-down approach is based on the data of 2,000,000 bank years. The computation of the risk measures is based on 2,000,000 simulated bank years for both approaches. In contrast to previous tables, the data generating process for the credit risk loss return only considers default risk, whereas the market risk loss return contains transition risk (without defaults), credit spread risk and interest rate risk. In the case of an increased PD, the default probability for initially Baa rated obligors is increased to from 0,185% to 1% and the other transition probabilities are proportionally adjusted. VaR_p : Value-at-Risk for the confidence level p, ES_p : Expected Shortfall for the confidence level p, $\hat{\rho}_{normal}^{\mathrm{CML}}$: estimate of the correlation parameter of the normal copula computed by canonical maximum likelihood (CML) estimation, Parameters: $\rho_R = 0.1$, $\rho_{X_r,S} = -0.2$, all other parameters are as described in section 4.1 or indicated in table 7.

		Bottom-Up Approach			Top-Down Approach		
initial rating Baa		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		8.72%	8.76%	9.05%	8.69%	8.71%	9.05%
standard deviation		25.89%	25.67%	25.31%	25.85%	25.43%	25.21%
	p						
VaR_p	0.99	75.82%	75.83%	74.11%	74.69%	74.94%	73.85%
	0.999	139.74%	142.49%	127.66%	130.12%	136.96%	125.58%
	0.9997	196.20%	191.08%	202.38%	172.26%	173.98%	186.33%
ES_p	0.99	110.32%	112.92%	107.54%	109.05%	109.59%	105.89%
	0.999	187.67%	197.86%	186.63%	187.91%	185.62%	178.32%
$\hat{\rho}_{normal}^{\mathrm{CML}}$					25.06%	20.94%	17.99%
initial rating Baa (increased PD)		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		12.98%	13.89%	13.85%	12.86%	13.84%	15.85%
standard deviation		29.12%	29.31%	28.38%	28.71%	28.56%	27.69%
	p						
VaR_p	0.99	91.86%	91.27%	87.04%	89.01%	88.12%	83.72%
	0.999	180.74%	182.92%	172.43%	163.55%	168.76%	152.45%
	0.9997	227.49%	287.73%	252.02%	229.41%	238.61%	208.37%
ES_p	0.99	141.00%	148.79%	138.01%	134.53%	137.07%	128.26%
	0.999	237.96%	280.90%	249.61%	234.74%	244.63%	221.87%
$\hat{\rho}_{normal}^{\mathrm{CML}}$					23.51%	20.92%	15.34%

Table 8:

Table 8 shows risk measures for the banking book loss return computed, on one hand, with the bottom-up approach and, on the other hand, with the top-down approach. For the top-down approach, a normal copula is used, and the empirical marginal distribution functions based on 2,000,000 bank years are employed. The estimation of the parameter of the top-down approach is based on the data of 2,000,000 bank years. The computation of the risk measures is based on 2,000,000 simulated bank years for both approaches. Credit risk losses are defined as losses due to rating transitions and defaults, but, in contrast to previous tables, market risk losses are defined as losses due to movements of the risk-free interest rates and due to deviations of the rating-specific credit spread from its mean in the rating class of the obligor at the end of a period. VaR_p : Value-at-Risk for the confidence level p, ES_p : Expected Shortfall for the confidence level p, $\hat{\rho}^{\text{CML}}_{normal}$: estimate of the correlation parameter of the normal copula computed by canonical maximum likelihood (CML) estimation, Parameters: $\rho_R = 0.1$, $\rho_{X_r,S} = -0.2$, all other parameters are as described in section 4.1 or indicated in table 8.

		Bottom-Up Approach			Top-Down Approach		
initial rating Aa		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		1.10%	1.17%	1.18%	1.10%	1.16%	1.17%
standard deviation		6.03%	5.77%	5.45%	6.03%	5.77%	5.45%
	p						
VaR_p	0.99	14.88%	14.01%	12.99%	14.94%	13.97%	13.00%
	0.999	21.27%	19.15%	17.67%	21.00%	19.06%	17.56%
	0.9997	23.75%	21.64%	19.73%	23.92%	21.51%	19.65%
ES_p	0.99	18.73%	17.50%	16.17%	18.68%	17.35%	16.15%
	0.999	24.50%	22.46%	20.53%	24.57%	22.14%	20.37%
$\hat{\rho}^{\text{CML}}_{normal}$					41.11%	-0.08%	-41.56%
initial rating Baa		$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$	$\rho_{X_r,R} = -0.2$	$\rho_{X_r,R} = 0$	$\rho_{X_r,R} = 0.2$
mean		8.72%	8.77%	9.05%	8.61%	8.66%	8.89%
standard deviation		25.89%	25.67%	25.31%	25.93%	25.30%	24.96%
	p						
VaR_p	0.99	75.65%	75.83%	74.11%	76.03%	74.41%	72.00%
	0.999	139.74%	142.49%	127.66%	132.97%	126.22%	120.58%
	0.9997	196.20%	191.08%	202.38%	178.00%	163.52%	154.00%
ES_p	0.99	110.34%	112.92%	107.55%	109.42%	105.70%	101.71%
	0.999	187.72%	197.84%	186.62%	184.27%	169.00%	160.29%
$\hat{\rho}^{\text{CML}}_{normal}$					6.70%	0.43%	-4.76%

Figures

Figure 1:

Figure 1 shows a typical cash flow of a banking book with positive term transformation that is used for the simulations.

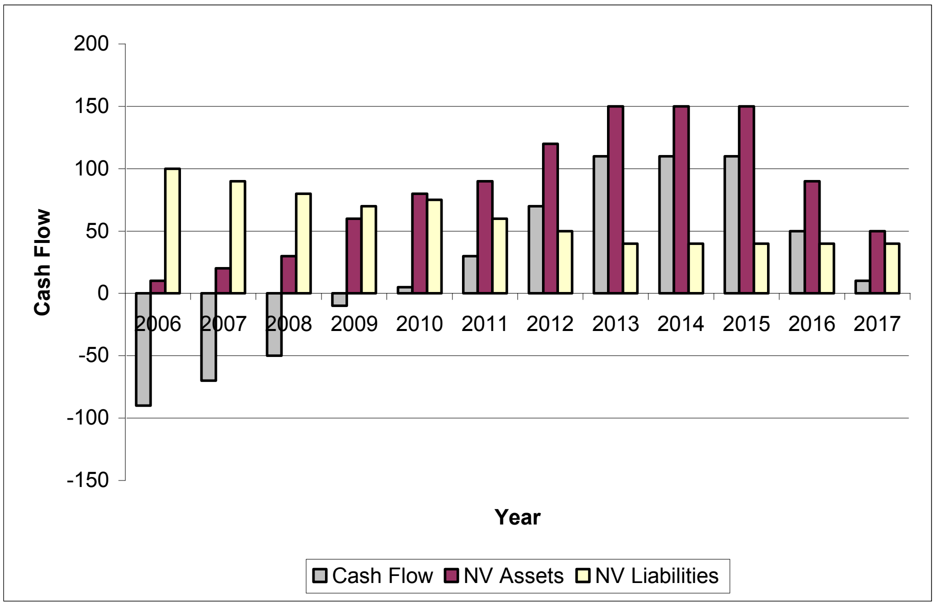

Figure 2:

Figure 2 shows the median p-values when the null hypothesis of a t-copula with varying degrees of freedom (dof) is tested with the GoF-test based on Rosenblatt's transformation. For this, 200 time series of 60 bank years are simulated. In case (a), the empirical marginal loss distributions based on 200,000 bank years are employed and the theoretical distribution of the AD test statistic is used. In case (b), the empirical marginal loss distributions are based on the simulated 60 bank years which are also used for estimating the correlation parameter of the t-copula and the bootstrap technique described in the Appendix is employed for simulating the distribution of the AD test statistic. The number of bootstrap simulations is 1,000. Parameters: $\rho_R = 0.1$, $\rho_{X_r,R} = -0.2$, $\rho_{X_r,S} = -0.2$.